www.mtg.in | January 2014 | ₹ 30

PRACTICE PAPERS

JEE Main & Advanced | Concept Boosters CLASS XI | CLASS XII

MATHEMATICS today

India's #1 MATHEMATICS MONTHLY for JEE (Main & Advanced)

CBSE Chapterwise

Class XII

Practice Paper West Bengal

Class XI

Enrich your concepts based on NCERT Pattern Limits, Derivatives,...

YOU ASKED WE ANSWERED

MATHS MUSING

10 Great Problems To Stimulate Creative & Critical Thinking

mtG Learning Media

Trust of more than 1 Crore Readers Since 1982

MATHEMATICS today

Vol. XXXII No. 1 January 2014

Corporate Office
Plot 99, Sector 44 Institutional Area, Gurgaon, (HR).
Tel : 0124-4951200
e-mail : info@mtg.in website : www.mtg.in

Regd. Office
406, Taj Apartment, Near Safdarjung Hospital,
Ring Road, New Delhi - 110 029.

Managing Editor : Mahabir Singh
Editor : Anil Ahlawat (BE, MBA)

CONTENTS

- **Maths Musing Problem Set - 133** 4
- **Practice Paper** 6
  JEE (Main & Advanced) 2014
- **Practice Paper** 16
  JEE (Main & Advanced) 2014
- **Mock Test Paper** 21
  WB JEE 2014
- **Concept Boosters (XI)** 27
- **Concept Boosters (XII)** 50
- **Enrich your Concepts** 68
  Based on NCERT Pattern (Class XI)
- **CBSE Board - 2014** 76
  Chapterwise Practice Paper (Series-9)
- **Maths Musing - Solutions** 87
- **You Asked, We Answered** 89

Owned, Printed and Published by Mahabir Singh from 406, Taj Apartment, New Delhi - 29 and printed by Personal Graphics and Advertisers (P) Ltd., Okhla Industrial Area, Phase-II, New Delhi. Readers are adviced to make appropriate thorough enquiries before acting upon any advertisements published in this magazine. Focus/Infocus features are marketing incentives MTG does not vouch or subscribe to the claims and representations made by advertisers. All disputes are subject to Delhi jurisdiction only.
Editor : Anil Ahlawat

Copyright© MTG Learning Media (P) Ltd.
All rights reserved. Reproduction in any form is prohibited.

editorial

Time to Sweeten Mathematics

It is a sad fact that the number of students disliking maths is quite large. If one is really keen to take literature, history or politics, it is a nice thing. But if they rush to those subjects because they do not have to study maths, it is a different thing. How many will take a medicine unless it is sweetened? The feeling, that more bitter the medicine, more it is effective, has disappeared. It is the manner of presentation that has to change. Mathematics is a growing science.

Let us see a small example. One studied vectors and scalars. Forces are vectors. We have new rules for addition. We call it vector mathematics. All of us know that energy is a scalar. In Einstein's theory of Relativity, even if a particle having mass is at rest, it has energy. One calls it rest mass energy. We can even call it 'Latent energy' by analogy with latent heat.

If the mass is at rest, with zero velocity, still it has energy, A. This is mc^2. If the energy B is added, the total energy, according to classical physics is $A + B$. According to Einstein, the total energy is not C, where $C = A + B$. It is $\sqrt{A^2 + B^2}$, But the concept of energy being a scalar has not changed.

mc^2, The total energy is $\sqrt{m_0^2c^4 + p^2c^2}$ where pc is the acquired energy.

This is not scalar mathematics and also it is not vector because energy is not a vector. This has hidden a lot of a truth which Einstein got as an inspiration. Boolean algebra used in computers is still more interesting.

How do we excite the students to higher energy levels?

We have to strictly follow the syllabus; but not in the same order. Every chapter should start from zero level to the ionization level where the student flies to higher levels by himself. The good teaching method is to walk and talk, talk and chalk, sprinkle with humour and ask questions to students at random. One should give a peek at the higher levels. Maths is more exciting than detective novels.

Anil Ahlawat
Editor

Subscribe online at **www.mtg.in**

Individual Subscription Rates	1 yr.	2 yrs.	3 yrs.	Combined Subscription Rates	1 yr.	2 yrs.	3 yrs.
Mathematics Today	300	500	675	PCM	800	1200	1700
Chemistry Today	300	500	675	PCB	800	1200	1700
Physics For You	300	500	675	PCMB	900	1500	2100
Biology Today	300	500	675				

Send D.D/M.O in favour of MTG Learning Media (P) Ltd.
Payments should be made directly to : MTG Learning Media (P) Ltd,
406, Taj Apartment, Near Safdarjung Hospital, Ring Road, New Delhi 29.
We have not appointed any subscription agent.

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala
Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 133

JEE MAIN

1. The number of pairs $(a, b), a < b$ of natural numbers, such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{2014}$, is

(a) 11 (b) 12 (c) 13 (d) 14

2. The point on the line $\frac{x+2}{2}=\frac{y+6}{3}=\frac{z-34}{-10}$ which is nearest to the line $\frac{x+6}{4}=\frac{y-7}{-3}=\frac{z-7}{-2}$ is (a, b, c) where $a + b + c =$

(a) 9 (b) 10 (c) 11 (d) 12

3. If $|\vec{a}|=1, |\vec{b}|=4, \vec{a}\cdot\vec{b}=2$ and $3\vec{b}+\vec{c}=2\vec{a}\times\vec{b}$, then the angle between $\vec{b}$ and $\vec{a}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

4. If the tangent at the point (a, b) on the curve $x^3 + y^3 = a^3 + b^3$ meets the curve again at the point (p, q), then

(a) $ap + bq + ab = 0$ (b) $bp + aq + ab = 0$
(c) $ap + bq - pq = 0$ (d) $bp + aq - pq = 0$

5. Let m be the number of 5-element subsets that can be chosen from the set of the first 15 natural numbers so that at least two of the five numbers are consecutive. The sum of the digits of m is

(a) 11 (b) 12 (c) 13 (d) 14

JEE ADVANCED

6. If a and b are natural numbers such that $a < b$ and H.M. of a and b is 2014, then $a =$

(a) 1060 (b) 1026 (c) 1368 (d) 1008

COMPREHENSION

A, B, C are points on the parabola $y^2 = 4x$ such that AC and BC are normals to the parabola.

7. The locus of the orthocentre of triangle ABC is

(a) $y^2 = x + 2$ (b) $x^2 = y + 2$
(c) $y^2 = x + 6$ (d) $x^2 = y + 6$

8. The locus of the circumcentre of triangle ABC is

(a) $y^2=\frac{1}{2}(x-1)$ (b) $y^2=\frac{1}{2}(x+1)$
(c) $y^2 = x - 1$ (d) $y^2 = x + 1$

INTEGER MATCH

9. If n is the number of right angled triangles with integer sides and inradius $r = 2014$, then the sum of the digits of n is

MATCHING LIST

10. Column I contains sums S and Column II consists of remainders when S is divided by 100.

Column-I		Column-II	
P.	$\sum_{n=1}^{2014} n\cdot i^{(n^2+n+2)} = S$	1.	9
Q.	$\sum_{n=1}^{2014} (2n-1)\cdot i^{(n^2+n+2)} = S$	2.	15
R.	$\sum_{n=1}^{2014} n^2\cdot i^{(n^2+n+2)} = S$	3.	28
S.	$\sum_{n=1}^{2014} (2n-1)^2\cdot i^{(n^2+n+2)} = S$	4.	78

	P	Q	R	S		P	Q	R	S
(a)	1	2	3	4	(b)	2	3	1	4
(c)	3	2	4	1	(d)	4	3	2	1

See Solution set of Maths Musing 132 on page no. 87

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

mtG

EXCEL IN CLASS XII BOARD EXAM

It is a matter of great satisfaction and encouragement to realise that our efforts in **"Excel in Physics, Excel in Chemistry** and **Excel in Biology"** have received overwhelming response. Nearly all the questions asked in the CBSE Board Examination 2013 were available in the books; fully solved. **'Excel in Mathematics'** is also on sale now.

We feel pleased to present the revised edition of the books. The books give comprehensive account of the subject according to the current syllabus and pattern of the CBSE Board Examination.

This will impart the students a clear and vivid understanding of the subject matter.

₹425 ₹425 ₹350 ₹450

Now on Sale!

FEATURES

- Comprehensive Theory
- Solutions to NCERT Questions
- Practice Questions for Board Exams
- Unit Test
- Latest CBSE Board Sample Papers (Solved)
- Previous CBSE Board Examination Papers (Solved)
- Model Test Papers

How to Order : Call your nearest book seller or contact us at 0124-4951200 or reader can purchase these books directly from us by sending demand draft or money order in favour of MTG Learning Media Pvt. Ltd., payable at New Delhi.

Buy online at **www.mtg.in**

Available at All leading bookshops

mtG BOOKS

For more information or to place your order:
Call at : 0124-4951200 @ Mail at : info@mtg.in
Log on to: www.mtg.in

PRACTICE PAPER 2014

JEE Main & Advanced

SECTION-I

SINGLE CORRECT ANSWER TYPE

1. Three numbers are chosen at random from 1, 2, ..., 30. What is the probability that they will form a G.P.?

(a) $\frac{12}{4060}$ (b) $\frac{10}{4060}$ (c) $\frac{19}{4060}$ (d) $\frac{20}{4060}$

2. If n balls are distributed into m boxes, so that each ball is equally likely to fall in any box, the probability that a specified box will contain r balls is

(a) $\frac{{}^{n}C_{r}(m-1)^{n-r}}{{}^{n}C_{m}}$ (b) $\frac{{}^{n}P_{r}(m-1)^{n-r}}{{}^{n}C_{m}}$

(c) $\frac{{}^{n}C_{r}(m-1)^{n-r}}{m^{n}}$ (d) $\frac{{}^{n}C_{r}(m-1)^{n-r}}{n^{m}}$

3. The probability that at least one of the events A and B occur is 0.4. If A and B be occur simultaneously with probability 0.1, then probability of A^C or B^C is equal to

(a) 1.2 (b) 1.4 (c) 1.5 (d) 1.8

4. In an attempt to land an unmanned rocket on the moon, the probability of a successful landing is 0.4 and the probability of monitoring system giving the correct information of landing is 0.9 in either case. The probability of a successful landing, it being known that the monitoring system indicated it correctly

(a) $\frac{2}{5}$ (b) $\frac{4}{25}$ (c) $\frac{9}{10}$ (d) $\frac{6}{7}$

5. Each of $N + 1$ identical urns marked 0, 1, 2, ..., N contains N balls so that the ith urn contains i black and $N - i$ white balls ($0 \le i \le N$). An urn is chosen at random and n balls are drawn from it one after another with replacement. If all the n balls turn out to be black, the probability that the next ball will also black [assume that N is large] is

(a) $\frac{n+1}{n+2}$ (b) $\frac{n-1}{n+1}$ (c) $\frac{n}{n+1}$ (d) $\frac{n}{n+2}$

6. The probability that a teacher will give a surprise test during any class meeting is 3/5. If a student is absent on two days, the probability that he will miss at least one test is

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) $\frac{20}{25}$ (d) $\frac{21}{25}$

7. If two independent events A and B are such that $P(A \cap B^C) = \frac{3}{25}$ and $P(A^C \cap B) = \frac{8}{25}$ and $P(A) > \frac{1}{2}$, the value of $P(A) + P(B)$ =

(a) $\frac{8}{5}$ (b) $\frac{4}{5}$

(c) $\frac{7}{5}$ (d) None of these

8. If two events A and B are such that $P(A^C) = 0.3$, $P(B) = 0.4$, $P(AB^C) = 0.5$, then $P(B/A \cup B^C)$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$ (d) $\frac{1}{2}$

9. The independent probabilities that A, B, C can solve the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ respectively. The probability that just two of them only solve the problem is

(a) $\frac{3}{4}$ (b) $\frac{2}{3}$

(c) $\frac{1}{4}$ (d) None of these

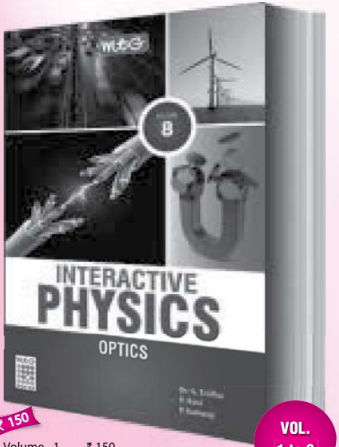
10. A bag A contains 2 white and 3 red balls and bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. The probability that it was drawn from bag B is

(a) $\frac{25}{52}$ (b) $\frac{13}{27}$ (c) $\frac{8}{17}$ (d) $\frac{25}{51}$

By : Sankar Ghosh, HOD (Math), Takshyashila. Mob : 09831244397.

After reading INTERACTIVE PHYSICS students have scored the highest marks in physics & succeeded in JEE (Main & Advanced) / PMT Exams.

INTERACTIVE PHYSICS



₹ 150

VOL. 1 to 8

Volume 1	₹ 150
Volume 2	₹ 170
Volume 3	₹ 200
Volume 4	₹ 150
Volume 5	₹ 175
Volume 6	₹ 200
Volume 7	₹ 160
Volume 8	₹ 150

HOW ?

- I.P. has given students the confidence to tackle the subject-*concepts have become crystal clear.*
- I.P. has improved problem solving skills-*student is able to solve nearly 90% problems of I.E Irodov and Resnick & Halliday himself.*
- Developed interest in *the subject-the student knows the subject so well that after reading I.P, physics has become his/her favourite subject.*
- A student who has read & understood the concepts found himself miles ahead of other competitors. *In exams like JEE (Main & Advanced) NEET/ PMTs top ranks were obtained because of good score in Physics*
- Average students have immensely gained from this book.
- No external help required - *I.P. develops the concepts so well, no tutor/teacher is required for mastering physics.*
- I.P. has laid the foundation so well that *physics will never be a problem even during higher education.*

Always Insist on MTG Books **Grab it from the nearest book-shop TODAY!**

A Book to Revolutionise Physics Learning and Teaching

Buy online at www.**mtg.in**

Available at leading bookshops throughout India.

MtG BOOKS

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

SECTON-II

MUTIPLE CORRECT ANSWER TYPE

11. A square is inscribed in a circle, if P_1 is the probability that a randomly selected point of a circle lies within the square and P_2 is the probability that the point lies outside the square, then

(a) $P_1 = P_2$ (b) $P_2 < P_1$

(c) $P_2 > P_1$ (d) $P_1^2 - P_2^2 < \frac{1}{3}$

12. A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them occurs is 1/3. The probability of occurrence of A is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

13. For two events A and B, $P(A \cap B)$ is

(a) $\geq P(A) + P(B) - 1$

(b) $\leq P(A) + P(B)$

(c) $P(A) + P(B) - P(A \cup B)$

(d) $P(A) + P(B) + P(A \cup B)$

14. An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i (i = 1, 2, 3) be the event that i^{th} digit of the number of ticket drawn is 1. Then

(a) A_1, A_2, A_3 are pairwise independent

(b) A_1, A_2, A_3 are pairwise not independent

(c) A_1, A_2, A_3 are mutually independent

(d) A_1, A_2, A_3 are not mutually independent

15. For any two events A and B

(a) $P(A/B) \geq \frac{P(A) + P(B) - 1}{P(B)}$

(b) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$ does not hold

(c) $P(A \cup B) = 1 - P(A^C) \cdot P(B^C)$ if A and B are independent

(d) $P(A \cup B) = 1 - P(A^C) \cdot P(B^C)$ if A and B are disjoint

SECTION-III

ASSERTION-REASON TYPE

Direction : (16-20) : Each question contains two statements. Statement-1 (Assertion) and Statement-2 (Reason). Each of the following questions also has four alternative choices, only one of which is correct answer. You have to select the correct choice.

(a) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation of statement-1

(b) Statement-1 is true, statement-2 is true, but statement-2 is not a correct explanation of statement-1

(c) Statement-1 is false, statement-2 is true

(d) Statement-1 is true, statement-2 is false

16. Statement-1 : The odds against A solving a certain problem are 4 to 3 and odds in favour of B solving the same problem are 7 to 5. The probability that the problem will be solved if they both try is $\frac{16}{21}$.

Statement-2 : For two independent events $P(A \cup B) = 1 - P(A^C) \cdot P(B^C)$.

17. Statement-1: In an examination 30% of the students failed in mathematics, 15% of the students failed in chemistry, and 10% failed in both mathematics and chemistry. A student is selected at random. The probability that he passed in mathematics, given that he failed in chemistry, is 2/3.

Statement-2 : For two events A and B, $P(A \cap B) = P(A) \cdot P(B/A)$, $P(A) > 0$, where $P(B/A)$ is the conditional probability of occurrence of B when the event A has already happened.

18. Statement-1 : A number is chosen at random from a set $S = \{1, 2, 3, ..., 6n + 3\}$. Let A and B are events defined as follows.

A = number is divisible by 2

B = number is divisible by 3

Then $A \times B$ are independent.

Statement-2 : If events A and B are independent then $P(A \cap B) = P(A)\ P(B)$.

19. Statement-1 : A box contains 4 white and 3 red balls, another box contains 3 white and 7 red balls. A box is chosen at random and a ball is drawn from it. The drawn ball is white. The probability that the ball drawn is from first box is $\frac{40}{61}$.

Statement-2 : If $E_1, E_2, ..., E_n$ are mutually disjoint events with $P(E_i) \neq 0$, (i = 1, 2, ..., n), then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$, such that $P(A) > 0$, we have $P(A) = \sum_{i=1}^{n} P(E_i)P(A/E_i)$.

20. Statement-1 : Let X be a continuous random variable with p.d.f given by

$$f(x) = \begin{cases} kx, & 0 \leq x < 1 \\ k, & 1 \leq x < 2 \\ -kx + 3k, & 2 \leq x < 3 \\ 0, & \text{elsewhere} \end{cases}$$

The value of k is $\frac{1}{2}$.

Statement-2 : Since $f(x)$ is the p.d.f. of x, therefore $\int_{-\infty}^{\infty} f(x)dx = 1$.

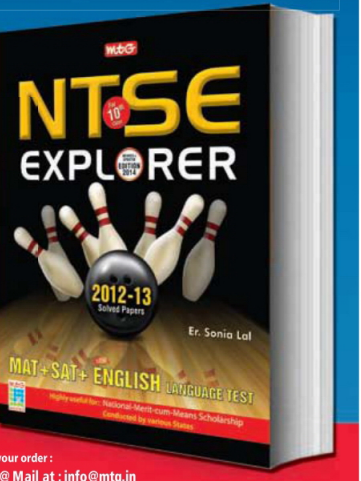
NTSE EXPLORER

Here's Presenting the most **Up-to-Date** and **Authentic Study Book** for National Talent Search Examination (NTSE).

For 10th Class

This book will help students to **master** the mandated NTSE curriculum and gain a **competitive edge** to excel in highly prestigious examinations as well as enrich their knowledge.

- An exhaustive account on Scholastic **Aptitude Test (SAT), Mental Ability Test (MAT) and English Language Test.**
- Detailed solutions of 2012-2013 NTSE papers.



₹ 575

Available at all leading bookshops throughout India.

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

SECTION-IV

INTEGER ANSWER TYPE

21. A student appears for test I, II and III. The student is successful if he passes either in test I and II or test I and III. The probabilities of the student passing in test I, II, III are x, y, $\frac{1}{2}$ respectively. If the probability that the student is successful is $\frac{1}{2}$, then $x + y$ is

22. The least number of times a fair coin must be tossed so that probability of getting at least one head is greater than 0.99 is

23. Out of odd number of tickets, numbered from 1 to the odd numbers, 3 tickets are drawn at random. The probability that the numbers on them are in A.P. is $\frac{21}{195}$. The number of tickets are

24. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the difference of number of outcomes that B must have so that A and B are independent, is

25. Assuming that half the population is vegetarian and each of 128 investigators take a sample of 10 individuals to see whether they are vegetarian, how many investigators would you expect to report 2 people or less are vegetarian?

SOLUTIONS

1. (c) : Let A be the event that the numbers of the triplets form a G.P.

Since 3 numbers can be selected from 30 numbers in ${}^{30}C_3$ ways, therefore total number of triplets = ${}^{30}C_3$

Now we count the triplets (arranged in increasing order) whose terms form a G.P. by listing them as follows.

Common ratio	Triplet
2	$\{(i, 2i, 4i), 1 \le i \le 7\}$
3	$\{(i, 3i, 9i), 1 \le i \le 3\}$
4	(1, 4, 16)
5	(1, 5, 25)
3/2	(4, 6, 9), (8, 12, 18), (12, 18, 27)
5/2	(4, 10, 25)
4/3	(9, 12, 16)
5/3	(9, 15, 25)
5/4	(16, 20, 25)

Thus among ${}^{30}C_3$ triplets, there are 19 triplets, whose terms form a G.P.

$\therefore$ Required probability $P(A) = \dfrac{19}{{}^{30}C_3} = \dfrac{19}{4060}$

2. (c) : The total number of ways of distributing n balls into m boxes is m^n which are assumed to be equally likely.

Since r balls which should go to the specified box can be chosen in ${}^nC_r$ ways and for any such way the remaining $(n - r)$ balls can be distributed to the $(m - 1)$ boxes in $(m-1)^{n-r}$ ways.

$\therefore$ The number of favourable cases = $\binom{n}{r}(m-1)^{n-r}$

$\therefore$ $P(A) = \dfrac{\binom{n}{r}(m-1)^{n-r}}{m^n}$

[where A is the event that a specified box will contain r balls]

3. (c) : Given that $P(A \cup B) = 0.4$ and $P(A \cap B) = 0.1$

$\therefore$ $P(A) + P(B) - P(A \cap B) = 0.4$

or $P(A) + P(B) = 0.4 + 0.1 = 0.5$

or $(1 - P(A^C) + 1 - P(B^C) = 0.5$

or $P(A^C) + P(B^C) = 2 - 0.5 = 1.5$

4. (d): Let A denote the event of a successful landing of the rocket, B_1 denote the event of the monitoring system indicating it correctly and B_2 denote the event of its indicating unsuccessful landing.

$\therefore$ $P(A) = 0.4$, $P(B_1/A) = 0.9$, $P(B_2/A^C) = 0.9$

The required probability $= P(A/B_1) = \dfrac{P(B_1 / A)\, P(A)}{P(B_1)}$

Now, $P(B_1) = P(A \cap B_1) + P(A^C \cap B_1)$

$= P(B_1/A)P(A) + P(B_1/A^C)\, P(A^C)$

$= P(B_1/A)P(A) + [1 - P(B_1{}^C/A^C)]\, P(A^C)$

$= P(B_1/A)\, P(A) + [1 - P(B_2/A^C)]\, P(A^C)$

$= 0.9 \times 0.4 + (1 - 0.9) \times 0.6 = 0.42$

Hence the required probability $= \dfrac{0.9 \times 0.4}{0.42} = \dfrac{6}{7}$

5. (a) : Let A_i be the event of choosing i^{th} urn and B the event of choosing n black balls in succession and C the event of drawing $(n + 1)^{\text{th}}$ ball as black. The required probability is

$$P(C/B) = \frac{P(B \cap C)}{P(B)}$$

Now, $P(B) = \sum_{i=0}^{N} P(A_i) \cdot P(B/A_i)$

$= \sum_{i=0}^{N} \frac{1}{N+1}\left(\frac{i}{n}\right)^n = \int_0^1 x^n dx$ when $N \to \infty$

Similarly,

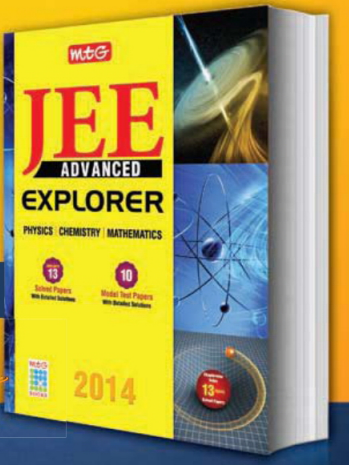
$P(B \cap C) = \sum_{i=0}^{N} \frac{1}{N+1}\left(\frac{i}{N}\right)^{n+1} = \int_0^1 x^{n+1} dx$ when $N \to \infty$

JEE ADVANCED EXPLORER

mtG

13 years Solved Papers with detailed Solutions

10 Model Test Papers



₹ 395

Buy online at www.mtg.in

Send DD/MO in favour of MTG Books. Available at all leading bookshops throughout the country.

mtG BOOKS

For more infromation or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

Hence the required probability

$$= \frac{\sum_{i=0}^{N}\left(\frac{i}{n}\right)^{n+1}}{\sum_{i=0}^{N}\left(\frac{i}{n}\right)^{n}} = \frac{\int_0^1 x^{n+1}dx}{\int_0^1 x^n dx} \quad \text{if } n \text{ is large}$$

$$= \frac{n+1}{n+2}$$

6. (d): Required probability

= 1 – *P* (no test is missed)

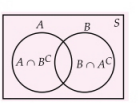
= 1 – *P* (no test on his first day of absence and no test of his second day of absence)

= 1 – *P* (no test of his first day of absence) × *P* (no test of his second day of absence)

$$= 1 - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{21}{25}$$

7. (c): Given that $P(A \cap B^C) = \frac{3}{25}$

and $P(A^C \cap B) = \frac{8}{25}$, $P(A) > \frac{1}{2}$



Let $P(A) = x$, $P(B) = y$

∴ We are given that

$P(A \cap B^C) = P(A) - P(A \cap B)$

$\frac{3}{25} = x - P(A)P(B)$

$\frac{3}{25} = x - xy$ [$\because A$ and B are independent]

Also, $P(A^C \cap B) = P(B) - P(A \cap B)$

$\Rightarrow \frac{8}{25} = y - xy$. Hence $y = x + \frac{1}{5}$

∴ Solving x and y we get $x = \frac{1}{5}$, $y = \frac{2}{5}$

or $x = \frac{3}{5}$ and $y = \frac{4}{5}$

As $P(A) > \frac{1}{2}$, we must have $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{5}$

∴ $P(A) + P(B) = \frac{7}{5}$

8. (a): $P(A^C) = 0.3$, $P(B) = 0.4$, $P(AB^C) = 0.5$

∴ $P(A \cup B^C) = P(A) + P(B^C) - P(A \cap B^C)$

$= 0.7 + 0.6 - 0.5 = 0.8$

Now, $P(B/A \cup B^C)$

$$= \frac{P[B \cap (A \cup B^C)]}{P(A \cup B^C)} = \frac{P[(B \cap A) \cup (B \cap B^C)]}{P(A \cup B^C)}$$

$$= \frac{P(A \cap B)}{P(A \cup B^C)} = \frac{P(A) - P(A \cap B^C)}{P(A \cup B^C)}$$

$$= \frac{0.7 - 0.5}{0.8} = \frac{0.2}{0.8} = \frac{1}{4}$$

9. (c): Given that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

∴ Probability that only two of them can solve the problem

$= P(A \cap B \cap C^C) + P(A \cap B^C \cap C) + P(A^C \cap B \cap C)$

$$= \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{6}{24} = \frac{1}{4}$$

10. (a): $P(E_1) = P(E_2) = \frac{1}{2}$

and $P(R/E_1) = \frac{3}{5}$, $P(R/E_2) = \frac{5}{9}$

By Bayes theorem

$$P(E_2/R) = \frac{P(E_2)\cdot P(R/E_2)}{P(E_1)\cdot P(R/E_1) + P(E_2)\cdot P(R/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5} + \frac{1}{2}\cdot\frac{5}{9}} = \frac{\frac{5}{9}}{\frac{3}{5} + \frac{5}{9}} = \frac{25}{27 + 25}$$

$$P(E_2/R) = \frac{25}{52}$$

11. (b, d): Let r be the radius of circle then area of inscribed square

$= \sqrt{2}r \times \sqrt{2}r = 2r^2$

∴ $P_1 = \frac{2r^2}{\pi r^2} = \frac{2}{\pi}$ ... (i)

$P_2 = 1 - P_1 = 1 - \frac{2}{\pi} = \frac{\pi - 2}{\pi}$ ... (ii)

∴ $P_2 < P_1$

Again, $P_1^2 - P_2^2 = (P_1 + P_2)(P_1 - P_2)$

$= 1 \cdot \frac{4 - \pi}{\pi} < \frac{1}{3}$ ($\because 3 < \pi < 4$)

12. (a, b): Let $P(A) = x$, $P(B) = y$

$P(A \cap B) = P(A)\,P(B) = xy = 1/6$ ... (i)

$P(A^C \cap B^C) = 1 - P(A \cup B)$

$P(A^C \cap B^C) = 1 - (x + y - xy)$

∴ $1 - (x + y - xy) = \frac{1}{3} \Rightarrow x + y = \frac{5}{6}$ ... (ii)

Solving (i) and (ii), we get

$x = \frac{1}{2}$, $y = \frac{1}{3}$ or $x = \frac{1}{3}$, $y = \frac{1}{2}$

13. (a,b, c): $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B)$ (option(c))

∴ $0 \le P(A \cup B) \le 1 \Rightarrow P(A \cap B) \le P(A) + P(B)$ (option (b))

and $P(A \cap B) \ge P(A) + P(B) - 1$ (option (a))

14. (a, d): Let A_1 is the event that the first digit of the number of the ticket drawn is 1 and the favourable cases for this are 112 and 121 *i.e.* two cases.

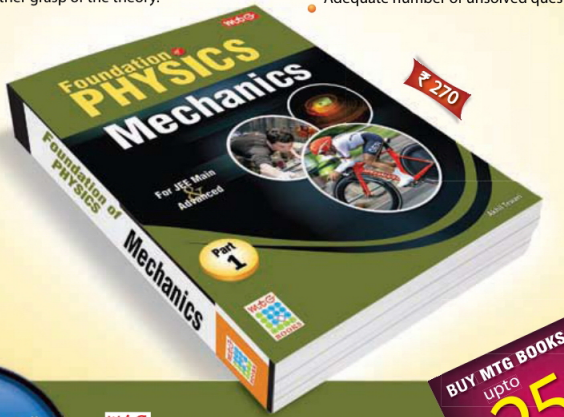
Foundation of PHYSICS

Mechanics

FOR JEE (MAIN & ADVANCED) AND OTHER ENGINEERING ENTRANCE EXAMS

Features of the Book

- Complete coverage of all the concepts in each unit followed by fundamental questions and examples for easy understanding.
- Adequate solved examples at the end of the chapter to enhance further grasp of the theory.
- Proper flow of the chapter has been maintained.
- The level of the questions vary from basic to tough keeping in mind the requirement of various entrance exams.
- Adequate number of unsolved questions.



Available at all leading bookshops throughout India

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

BUY MTG BOOKS ONLINE upto 25% OFF
www.mtg.in

$\therefore \quad P(A_1) = \frac{2}{4} = \frac{1}{2}.$

Similarly $P(A_2) = P(A_3) = \frac{2}{4} = \frac{1}{2}$

$A_1 \cap A_2$ is the event that the first two digits in the number which the selected tickets bear are each equal to unity and the only favourable case is ticket with number 112.

$\therefore \quad P(A_1 \cap A_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A_1) \cdot P(A_2)$

Similarly, $P(A_2 \cap A_3) = \frac{1}{4} = P(A_2) \cdot P(A_3)$

and $P(A_3 \cap A_1) = \frac{1}{4} = P(A_3) \cdot P(A_1)$

Thus we conclude that the events A_1, A_2 and A_3 are pairwise independent.

Now, $P(A_1 \cap A_2 \cap A_3)$

$= P$ {all the three digits in the number selected are 1's}

$= P(\phi) = 0$

$\neq P(A_1)\, P(A_2)\, P(A_3)$

Hence A_1, A_2 and A_3 are though pairwise but not mutually independent.

15. (a, c) : $P(A / B) = \dfrac{P(A \cap B)}{P(B)}$

$$\geq \frac{P(A) + P(B) - 1}{P(B)}$$

Also, $P(A \cup B) = P(A^C \cap B^C)^C = 1 - P(A^C \cap B^C)$

$= 1 - P(A^C)\, P(B^C)$ [If A, B are independent]

16. (a) : $P(A) = \frac{3}{7}, \quad P(B) = \frac{7}{12}$

Probability that problem will be solved, if at least one of them can solve the problem,

i.e. $P(A \cup B) = 1 - P(A \cup B)^C = 1 - P(A^C \cap B^C)$

$= 1 - P(A^C)\, P(B^C) = 1 - \frac{4}{7} \cdot \frac{5}{12} = \frac{84 - 20}{84} = \frac{64}{84} = \frac{16}{21}$

Thus option (a) is correct answer.

17. (c) : Given that $P(M) = \frac{3}{10}, \; P(C) = \frac{15}{100} = \frac{3}{20}$

$P(M \cap C) = \frac{1}{10}$

Required probability $P(M^C / C) = \dfrac{P(M^C \cap C)}{P(C)}$

$$= \frac{P(C) - P(M \cap C)}{P(C)} = \frac{\frac{3}{20} - \frac{1}{10}}{\frac{3}{20}} = \frac{1}{3}$$

Thus the option (c) is the correct answer.

18. (c) : A is divisible by 2 $\quad \therefore \quad A = \{2, 4, 6, ..., 6n + 2\}$

B is divisible by 3 $\quad \therefore \quad B = \{3, 6, 9, ..., 6n + 3\}$

and $A \cap B = \{6, 12, 18, ..., 6n\}$

Let the number of terms in set A be N.

$\therefore \quad 2 + (N - 1) \times 2 = 6n + 2 \;\Rightarrow\; N = 3n + 1$

Similarly, the number of terms in set B is $2n + 1$ and in $A \cap B$ is n

$\therefore \quad P(A) = \frac{3n+1}{6n+3}; \quad P(B) = \frac{2n+1}{3(2n+1)} = \frac{1}{3}$

and $P(A \cap B) = \frac{n}{3(2n+1)} = \frac{1}{3} \cdot \frac{n}{2n+1}$

$P(A \cap B) \neq P(A)\, P(B)$

$\Rightarrow$ A and B are not independent

Hence option (c) is correct answer.

19. (b): Let E_1 and E_2 denote the events that bag I and bag II is chosen respectively, and let A be the event that the ball taken from the selected bag is white.

Then $P(E_1) = P(E_2) = \frac{1}{2}$

$P(A / E_1) = \frac{{}^4C_1}{{}^7C_1} = \frac{4}{7}, \; P(A / E_2) = \frac{{}^3C_1}{{}^{10}C_1} = \frac{3}{10}$

$$\therefore \quad P(E_1 / A) = \frac{P(E_1) \cdot P(A / E_1)}{\sum_{i=1}^{2} P(E_i) P(A / E_i)}$$

$$= \frac{P(E_1) \cdot P(A / E_1)}{P(E_1) \cdot P(A / E_1) + P(E_2) \cdot P(A / E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{4}{7}}{\frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{3}{10}} = \frac{\frac{4}{7}}{\frac{4}{7} + \frac{3}{10}} = \frac{\frac{4}{7}}{\frac{61}{70}} = \frac{4}{7} \times \frac{70}{61} = \frac{40}{61}$$

Thus the statement-1 is true, statement-2 is also true but statement-2 is not the proper explanation of statement-1.

20. (a) : Since $f(x)$ is a p.d.f.

$$\therefore \quad \int_0^3 f(x)\, dx = 1$$

$$\Rightarrow \quad \int_0^1 kx\, dx + \int_1^2 k\, dx + \int_2^3 (-kx + 3k) dx = 1$$

$$\Rightarrow \quad k\left(\frac{1}{2} - 0\right) + k(2 - 1) + 3k(3 - 2) - \frac{k}{2}(9 - 4) = 1$$

$$\Rightarrow \quad \frac{k}{2} + k + 3k - \frac{5k}{2} = 1 \;\Rightarrow\; 4k - 2k = 1$$

$$\Rightarrow \quad 2k = 1 \;\Rightarrow\; k = \frac{1}{2}$$

Thus the option (a) is correct.

21. (1) : Given that $P(\text{I}) = x$, $P(\text{II}) = y$ and $P(\text{III}) = \frac{1}{2}$

According to the given conditions,

$\therefore \quad P[(\text{I} \cap \text{II}) \cup (\text{I} \cap \text{III}] = \frac{1}{2}$

or, $P(\text{I} \cap \text{II}) + P(\text{I} \cap \text{III}) - P(\text{I} \cap \text{II} \cap \text{III}) = \frac{1}{2}$

or, $P(\text{I})\,P(\text{II}) + P(\text{I})\,P(\text{III}) - P(\text{I})\,P(\text{II})\,P(\text{III}) = \frac{1}{2}$

or, $xy + x \cdot \frac{1}{2} - xy \cdot \frac{1}{2} = \frac{1}{2}$

or, $\frac{1}{2}xy + \frac{1}{2}x = \frac{1}{2}$ or $x(1+y) = 1$

The above equation is satisfied by only $x = 1$ and $y = 0$
Thus, $x + y = 1 + 0 = 1$

22. (7) : Let n be the no. of toss of the fair coin.

$\therefore$ The probability of getting at least one head ≥ 0.99

$\therefore$ Probability of getting no head ≤ 0.01

$\Rightarrow \quad \frac{1}{2^n} \leq \frac{1}{100} \quad \Rightarrow \quad 2^n \geq 100 \quad \Rightarrow \quad n = 7$

23. (7) : Let the number of tickets be $2n + 1$
Since 3 tickets can be drawn from $2n + 1$ in ${}^{2n+1}C_3$ ways, therefore the number of exhaustive cases $= {}^{2n+1}C_3$

No. of favourable cases $= n^2$
Let A be the event that the numbers on the drawn ticket are in A.P.

$\therefore \quad P(A) = \frac{n^2}{{}^{2n}C_3} = \frac{3n}{4n^2 - 1}$

But $\frac{3n}{4n^2 - 1} = \frac{21}{195}$. $\therefore \quad n = 7$

24. (5) : $P(A) = \frac{4}{10}$, if $P(B) = \frac{x}{10}$, then

$P(A \cap B) = P(A)\,P(B)$

or $P(A \cap B) = \frac{4}{10} \cdot \frac{x}{10} = \frac{x}{25} = \frac{2x/5}{10}$

$\therefore \quad \frac{2x}{5}$ is an integer, $x = 5$ or 10

$\therefore$ The difference of number of outcomes that B must have $(10 - 5) = 5$.

25. (7) : Let X be the random variable that the selected person is vegetarian.

$\therefore \quad P(X = x) = \binom{10}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10}$

Now $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= \frac{1}{2^{10}}\left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2}\right] = \frac{1}{2^{10}}(1 + 10 + 45) = \frac{56}{1024}$

The number of investigators, those who will give report that 2 people or less are vegetarian

$= \frac{56 \times 128}{1024} = 7$

ATTENTION COACHING INSTITUTES:
a great offer from MTG

MTG offers **"Classroom Study Material"** for JEE (Main & Advanced), AIPMT and FOUNDATION MATERIAL for Class 8, 9, 10, 11 & 12 **with YOUR BRAND NAME & COVER DESIGN**.

This study material will save you lots of money spent on teachers, typing, proof-reading and printing. Also, you will save enormous time. Normally, a good study material takes 2 years to develop. But you can have the material printed with your logo delivered at your doorstep.

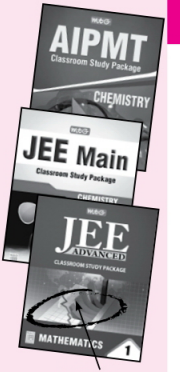
Profit from associating with MTG Brand – the most popular name in educational publishing for JEE (Main & Advanced)/AIPMT/PMT ....

Order sample chapters on Phone/Fax/e-mail.

Phone : 0124-4951200, 09312680856
e-mail : sales@mtg.in | www.mtg.in

★ EXCELLENT QUALITY ★
★ CONTENT ★ PAPER ★ PRINTING

CLASSROOM STUDY MATERIAL



Your logo here

PRACTICE PAPER 2014

JEE (Main & Advanced) and other PETs

1. The set of equations
$\lambda x - y + (\cos\theta)z = 0$
$3x + y + 2z = 0$
$(\cos\theta)x + y + 2z = 0$
$0 \le \theta < 2\pi$, has non–trivial solutions
(a) for no value of λ and θ
(b) for all values of λ and θ
(c) for all values of λ and only two values of θ
(d) for only one value of λ and all values of θ.

2. Orthogonal trajectories of family of parabolas $y^2 = 4a(x + a)$ where 'a' is an arbitrary constant is
(a) $ax^2 = 4cy$ (b) $x^2 + y^2 = a^2$
(c) $y = ce^{-\frac{x}{2a}}$
(d) $axy = c^2$, where c is a constant.

3. If $\frac{5z_1}{7z_2}$ is purely imaginary, then $\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| =$
(a) 5/7 (b) 7/5 (c) 25/29 (d) 1

4. The mean daily profit made by a shopkeeper in a month of 30 days was ₹ 350. If the mean profit for the first twenty days was ₹ 400, then the mean profit for the last 10 days would be
(a) ₹ 200 (b) ₹ 250 (c) ₹ 800 (d) ₹ 300

5. Consider points $A(3, 4)$ and $B(7, 13)$. If P be a point on the line $y = x$ such that $PA + PB$ is minimum, then coordinates of P are
(a) $\left(\frac{12}{7}, \frac{12}{7}\right)$ (b) $\left(\frac{13}{7}, \frac{13}{7}\right)$
(c) $\left(\frac{31}{7}, \frac{31}{7}\right)$ (d) $(0, 0)$

6. $\lim_{x\to 0} \frac{\log(2 + x^2) - \log(2 - x^2)}{x^2} = k$, the value of k is
(a) –1 (b) 2 (c) 1 (d) 0

7. The sum of the series $\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + ...\infty$ is equal to
(a) 1 (b) 0 (c) –1 (d) 2

8. If $f : R \to R$, $f(x + y) = f(x) \cdot f(y)$. If $f(1) = 1$, then $\sum_{k=1}^{n} f(k) =$
(a) n (b) $n(n + 1)(2n + 1)/6$
(c) 1 (d) $n(n + 1)/2$

9. If $f(x) = \begin{cases} \frac{\sin(a+1)x + \sin x}{x}, & \text{for } x < 0 \\ c, & \text{for } x = 0 \\ \frac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, & \text{for } x > 0 \end{cases}$
is continuous at $x = 0$, then $a =$
(a) 3/2 (b) –3/2 (c) 1/4 (d) –1/4

10. If $f(x) = x^n$, then the value of
$\{f(1)\}^2 + \left\{\frac{f'(1)}{1!}\right\}^2 + \left\{\frac{f''(1)}{2!}\right\}^2 + ... + \left\{\frac{f^n(1)}{n!}\right\}^2$
is given by
(a) ${}^{2n}C_n$ (b) ${}^{2n}C_{n-1}$ (c) ${}^{2n}C_{n+1}$ (d) ${}^{2n}C_1$

11. For the three events A, B and C, P(exactly one of the events A or B occurs) = P (exactly one of the events B or C occurs) = P(exactly one of the events C or A occurs) = p and P(all the three events occurs simultaneously) = p^2, where $0 < p < 1/2$. Then the probability of at least one of the three events A, B and C occurring is
(a) $\frac{1}{2}(3p + 2p^2)$ (b) $\frac{1}{4}(p + 3p^2)$
(c) $\frac{1}{2}(p + 3p^2)$ (d) $\frac{1}{4}(3p + 2p^2)$

12. The area of the smaller region bounded by the curves $x^2 + y^2 = 5$ and $y^2 = 4x$ is
(a) $\frac{\pi}{4} + \frac{1}{6}$ (b) $\frac{\pi}{4} - \frac{1}{6}$
(c) $2\left(\frac{1}{3} - \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)$ (d) $2\left(\frac{1}{3} + \frac{5}{2}\sin^{-1}\frac{2}{\sqrt{5}}\right)$

13. If a, b, c are real, then $f(x) = \begin{vmatrix} x + a^2 & ab & ac \\ ab & x + b^2 & bc \\ ac & bc & x + c^2 \end{vmatrix}$ is decreasing in

By : **Vidyalankar Institute**, Pearl Centre, Senapati Bapat Marg, Dadar (W), Mumbai - 28. Tel.: (022) 24306367

(a) $\left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$ (b) $\left(0, \frac{2}{3}(a^2+b^2+c^2)\right)$

(c) $\left(\frac{a^2+b^2+c^2}{3}, 0\right)$ (d) None of these

14. This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement-1 : The equation $ax^2 + bx + c = 0$ cannot have rational roots if a, b, c are odd integers.

Statement-2 : If an odd number does not leave remainder 1 when divided by 8, then it cannot be a perfect square.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

15. This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement-1 : The equation of a line through the point (5, 7) and parallel to the line

$$y = \lim_{t\to 0} \frac{e^{xt}-1}{t} + 7 \text{ is } x - y + 2 = 0.$$

Statement-2 : $\lim_{t\to 0} \frac{e^{xt}-1}{t} = x$, by expansion or by L'Hospital's rule.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

16. This question contains Statement-1 and Statement-2. Of the four choices given after the statements, choose the one that best describes the two statements.

Statement – 1 : The expression $\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \ldots$ attains maximum value when $r = 100$.

Statement-2 : $\binom{2n}{r}$ is maximum when $r = n$.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false.
(d) Statement-1 is false, Statement-2 is true.

17. If $P(n) : n^2 + n$ is an odd integer. It is seen that truth of $P(n) \Rightarrow$ the truth of $P(n+1)$. Therefore, $P(n)$ is true for all

(a) $n > 1$ (b) n
(c) $n > 2$ (d) none of these

18. If $y = \sqrt{\cos x^2 + \sqrt{\cos x^2 + \sqrt{\cos x^2 + \ldots \text{ to } \infty}}}$, then $\frac{dy}{dx}$ is

(a) $\frac{-\sin x^2}{x(2y-1)}$ (b) $\frac{-2x\sin x^2}{2y-1}$

(c) $\frac{-\sin x}{2y-1}$ (d) none of these

19. Area lying between the curves $y^2 = 4x$ and $y = 2x$ is

(a) 2/3 (b) 1/3
(c) 1/4 (d) none of these

20. If $f(x)$ is a continuous function satisfying $f(x)f(1/x) = f(x) + f(1/x)$ and $f(1) > 0$, then $\lim_{x\to 1} f(x)$ is equal to

(a) 2 (b) 1
(c) 3 (d) none of these

21. For any integer n, the integral $\int_0^{\pi} e^{\cos^2 x} \cos^3 x(2n+1)x\,dx$ has the value

(a) π (b) 1
(c) 0 (d) none of these

22. Let E and F be two independent events. The probability that both E and F happen is 1/12 and the probability that neither E nor F happens is 1/2. Then

(a) $P(E) = 1/3, P(F) = 1/4$
(b) $P(E) = 1/2, P(F) = 1/6$
(c) $P(E) = 1/6, P(F) = 1/2$
(d) $P(E) = 1/8, P(F) = 1/3$

23. If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^nC_0 + \sum_{k=0}^{n-1} {}^{n+k}C_{k+1}$ equals

(a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_{n+1}$
(c) ${}^{2n}C_n$ (d) ${}^{n}C_{n-1}$

24. The number of integral terms in the expansion of $(\sqrt{7}+\sqrt[6]{5})^{78}$ is
(a) 13 (b) 14 (c) 16 (d) 0

25. Let $f(x)$ be a function satisfying the condition $f'(x)=\sqrt{1-(f(x))^2}$ with $f(0)=0$, then the value of the integral $\int e^x f^{-1}(x)dx+\int\frac{e^x}{\sqrt{1-x^2}}dx$ is
(a) $e^x\cos x+C$ (b) $e^x\sin x+e^x\sin^{-1}x+C$
(c) $e^x\sin^{-1}x+C$ (d) $e^x\cos^{-1}x+e^x\sin^{-1}x+C$

26. Algebraic sum of the intercepts made by the plane $x+3y-4z+6=0$ on the axes is
(a) –13/2 (b) 19/2 (c) –22/3 (d) 26/3

27. If the roots of $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal, then a, b, c are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

PARAGRAPH TYPE

This section contains 1 paragraph. Based upon the paragraph 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

$P: y^2=8x,\ E: \frac{x^2}{4}+\frac{y^2}{15}=1$

28. Equation of a tangent common to both the parabola P and the ellipse E is
(a) $x-2y+8=0$ (b) $2x-y+8=0$
(c) $x+2y-8=0$ (d) $2x-y-8=0$

29. Equation of the normal at the point of contact of the common tangent, which makes an acute angle with the positive direction of x-axis, to the parabola P is
(a) $2x+y=24$ (b) $2x+y+24=0$
(c) $2x+y=48$ (d) $2x+y+48=0$

30. Point of contact of a common tangent to P and E on the ellipse is
(a) (1/2, 15/4) (b) (–1/2, 15/4)
(c) (1/2, –15/2) (d) (–1/2, –15/2)

SOLUTIONS

1. (a): Determinant of coefficients

$$=\begin{vmatrix}\lambda & -1 & \cos\theta\\ 3 & 1 & 2\\ \cos\theta & 1 & 2\end{vmatrix}=\cos\theta-\cos^2\theta+6$$

and this is positive for all θ since $|\cos\theta|\le 1$. The only solution is therefore the trivial solution.

2. (c): $2y\cdot\frac{dy}{dx}=4a \Rightarrow \frac{dy}{dx}=\frac{2a}{y}$

For orthogonal trajectory,

$$-\frac{dx}{dy}=\frac{2a}{y} \Rightarrow \int\frac{dy}{y}=\int-\frac{1}{2a}dx$$

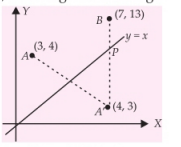
$$\Rightarrow \ln y=-\frac{x}{2a}+k \Rightarrow y=ce^{-\frac{x}{2a}}$$

3. (d): Let $\frac{5z_1}{7z_2}=Ki\ (K\in R)$, then $\frac{z_1}{z_2}=\left(\frac{7K}{5}\right)i$

Consider $\left|\frac{2z_1+3z_2}{2z_1-3z_2}\right|=\left|\frac{2\left(\frac{z_1}{z_2}\right)+3}{2\left(\frac{z_1}{z_2}\right)-3}\right|=\left|\frac{\frac{14K}{5}i+3}{\frac{14K}{5}i-3}\right|=1$

4. (b): $30\times 350=20\times 400+10\times x$
$\Rightarrow 10x=10500-8000 \Rightarrow 10x=2500 \Rightarrow x=250$

5. (c): Consider a point A', the image of A through $y=x$



$\therefore$ Coordinates of $A'=(4,3)$

[Notice that A and B lie to the same side with respect to $y=x$].

Then $PA=PA'$.

Thus, $PA+PB$ is minimum, if $PA'+PB$ is minimum, if P, A', B are collinerar.

Now,

AB is $y-3=\frac{13-3}{7-4}(x-4) \Rightarrow 3y-10x+31=0$

It intersects $y=x$ at $\left(\frac{31}{7},\frac{31}{7}\right)$, which is the required point P.

6. (c): $\lim_{x\to 0}\frac{1}{x^2}[\log(2+x^2)-\log(2-x^2)]$

$$=\lim_{x\to 0}\frac{1}{x^2}\left[\log\left(\frac{2+x^2}{2}\right)-\log\left(\frac{2-x^2}{2}\right)\right]$$

$$=\lim_{x\to 0}\left[\log\left(1+\frac{x^2}{2}\right)^{1/x^2}-\log\left(1-\frac{x^2}{2}\right)^{1/x^2}\right]$$

$$=\log e^{1/2}-\log e^{-1/2}=\frac{1}{2}+\frac{1}{2}=1$$

7. (a): The given series

$$\sum_{r=1}^{\infty}\frac{(2r+1)}{r^2(r+1)^2}=\sum_{r=1}^{\infty}\frac{1}{r^2}-\sum_{r=1}^{\infty}\frac{1}{(r+1)^2}=1$$

8. (a): $f(1)=1 \Rightarrow f(2)=1 \Rightarrow f(3)=1$

$f(r)=1\ \forall\ r=1, 2, ..., n$

$\sum_{k=1}^{n}f(k)=n$

9. (b): $f(0-0)=\lim_{h\to 0} f(0-h)$

$=\lim_{h\to 0}\frac{\sin(a+1)(-h)+\sin(-h)}{0-h}=\lim_{h\to 0}\frac{(a+1+1)(-h)}{(-h)}=a+2$

$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\frac{(h+bh^2)^{1/2}-h^{1/2}}{bh^{3/2}}$

$=\lim_{h\to 0}\frac{b^{1/2}\left[1+\frac{bh}{2}-1\right]}{bh\cdot h^{1/2}}=\frac{1}{2}$

$\therefore$ The given function is continuous $f(0-0)=f(0+0)$.

$\therefore \quad a+2=\frac{1}{2},\ a=-\frac{3}{2}$

10. (a): $f(x)=x^n \Rightarrow \frac{f^r(1)}{r!}={}^nC_r$

$\sum_{r=0}^{n}({}^nC_r)^2={}^{2n}C_n$

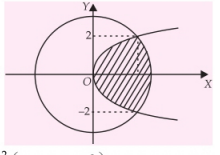
11. (a): $P(A\cup B\cup C)=\sum P(A)-P(AB)-P(BC)$
$-P(AC)+P(A\cap B\cap C)$

$=\frac{1}{2}\begin{Bmatrix}P(A)-P(AB)+P(B)-P(BC)+P(B)\\-P(AB)+P(C)-P(BC)+P(A)-P(AC)\\+P(C)-P(AC)\end{Bmatrix}+P(ABC)$

$=\frac{1}{2}\left\{P(A\bar{B})+P(\bar{A}B)+P(A\bar{C})+P(B\bar{C})+P(\bar{B}C)+P(\bar{A}C)\right\}$
$+P(ABC)$

$=\frac{1}{2}(3p)+p^2=\frac{1}{2}(3p+2p^2)$

12. (d):



Area $=\int_{-2}^{2}\left(\sqrt{5-y^2}-\frac{y^2}{4}\right)dy$

$=2\int_{0}^{2}\left(\sqrt{5-y^2}-\frac{y^2}{4}\right)dy=2\left(\frac{1}{3}+\frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

13. (a): $f'(x)=\begin{vmatrix}1&0&0\\ab&x+b^2&bc\\ac&bc&x+c^2\end{vmatrix}+\begin{vmatrix}x+a^2&ab&ac\\0&1&0\\ac&bc&x+c^2\end{vmatrix}$

$+\begin{vmatrix}x+a^2&ab&ac\\ab&x+b^2&bc\\0&0&1\end{vmatrix}$

$=(x+b^2)(x+c^2)-b^2c^2+(x+a^2)(x+c^2)-a^2c^2$
$+(x+a^2)(x+b^2)-a^2b^2$

$=3x^2+2x(a^2+b^2+c^2)$

$f(x)$ will be decreasing when $f'(x)<0$

$\Rightarrow \quad 3x^2+2x(a^2+b^2+c^2)<0$

$\Rightarrow \quad x\in\left(-\frac{2}{3}(a^2+b^2+c^2),\ 0\right)$

14. (a): The statement-2 is true since the square of an odd number $2l+1$ is given by

$(2l+1)^2=4l^2+4l+1=4l(l+1)+1$

$=8k+1\ (\because\ l(l+1)$ is a multiple of 2)

$\Rightarrow$ Square of odd number leaves remainder 1 when divided by 8

Thus Statement-1 is true if all the coefficients are odd.

Let $a=2l+1,\ b=2m+1,\ c=2n+1$

then $b^2-4ac=(2m+1)^2-4(2l+1)(2n+1)$

$=4m^2+4m-16ln-8l-8n-3$

$=8\left[\frac{m(m+1)}{2}-2\ln-l-n\right]-3$

$=8k-3\quad\left[\because\ \frac{m(m+1)}{2}\text{ is an integer}\right]$

$\Rightarrow$ b^2-4ac is an odd number which cannot be a perfect square.

$\Rightarrow \quad \text{roots}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{\text{rational}\pm\text{irrational}}{\text{rational}}$
$=\text{irrational}$

$\Rightarrow$ statement-1 is true.

15. (a): $y=x+\lambda$, pass through (5, 7)

$\therefore \quad \lambda=2.$

16. (d): We have

$$\frac{\binom{2n}{r}}{\binom{2n}{r+1}}=\frac{(2n)!}{r!(2n-r)!}\cdot\frac{(r+1)!(2n-r-1)!}{(2n)!}=\frac{r+1}{2n-r}$$

Since for $0\le r\le n-1,\ \frac{r+1}{2n-r}<1$, we get

$$\binom{2n}{0}<\binom{2n}{1}<\ldots.<\binom{2n}{n+1}<\binom{2n}{n}$$

Also, as $\binom{2n}{r}=\binom{2n}{2n-r}$

$$\binom{2n}{2n}<\binom{2n}{2n-1}<\ldots.<\binom{2n}{n+1}<\binom{2n}{n}$$

Thus, $\binom{2n}{r}$ is maximum when $r=n$.

Next, $\binom{40}{r}\binom{60}{0}+\binom{40}{r-1}\binom{40}{1}+\ldots$

= the number ways of selecting r persons out of 40 men and 60 women

$= \begin{pmatrix} 100 \\ r \end{pmatrix}$ which is maximum when $r = 50$.

17. (d): The sufficient condition for the statement : $P(n)$ to be true for $n \geq a$ is

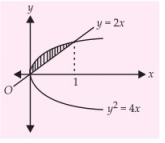
(i) truth of $P(n) \Rightarrow$ truth of $P(n+1)$

(ii) $P(a)$ is true. Hence answer is none of these.

18. (b): $y = \sqrt{\cos x^2 + y} \Rightarrow y^2 - y = \cos x^2$

$\therefore \quad 2y\dfrac{dy}{dx} - \dfrac{dy}{dx} = -\sin x^2 \cdot 2x \Rightarrow \dfrac{dy}{dx} = -\dfrac{2x\sin x^2}{2y-1}$

19. (b):



$(2x)^2 = 4x \Rightarrow 4x^2 = 4x$

$\Rightarrow \quad 4x^2 - 4x = 0 \quad \Rightarrow 4x(x-1) = 0$

$x = 0, 1$

Required area $= \displaystyle\int_0^1 (2\sqrt{x} - 2x)dx$

$$= 2\cdot\frac{2}{3}x^{3/2} - x^2\Big|_0^1 = \frac{4}{3} - 1 = \frac{1}{3}$$

20. (a): Since f is continuous function so,

$\lim\limits_{x\to 1} f(x) = f(1)$.

Put $x = 1$ in the given equation we have

$(f(1))^2 = 2f(1)$, so $f(1) = 0$ or 2.

Since $f(1) > 0$, so $f(1) = 2$.

21. (c)

22. (a): Let $P(E) = x$, $P(F) = y$, $xy = \dfrac{1}{12}$

$(1-x)(1-y) = \dfrac{1}{2}$, or $1 - (x+y) + xy = \dfrac{1}{2}$

$x + y = \dfrac{1}{2} + \dfrac{1}{12} = \dfrac{7}{12}$; $\quad x + y = \dfrac{7}{12}$, $\quad xy = \dfrac{1}{12}$

$P(E) = \dfrac{1}{3}$ and $P(F) = \dfrac{1}{4}$

or, $\quad P(E) = \dfrac{1}{4}$ and $P(F) = \dfrac{1}{3}$

23. (c): ${}^nC_0 + \displaystyle\sum_{k=0}^{n-1} {}^{n+k}C_{k+1} = ({}^nC_0 + {}^nC_1) + {}^{n+1}C_2$

$+ {}^{n+2}C_3 + ... + {}^{n+n-1}C_n$

$= ({}^{n+1}C_1 + {}^{n+1}C_2) + {}^{n+2}C_3 + ... + {}^{2n-1}C_n$

$= ({}^{n+2}C_2 + {}^{n+2}C_3) + ... + {}^{2n-1}C_n$

Similarly, ${}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n$

24. (b): k^{th} term will be ${}^{78}C_k(\sqrt{7})^k(\sqrt[6]{5})^{78-k}$.

It is integral if $78 - k$ is a multiple of 6 and k is even. There will be 14 such multiple from 0 to 78.

25. (c): $f'(x) = \sqrt{1-\{f(x)\}^2}$

$\Rightarrow \quad \dfrac{df(x)}{\sqrt{1-\{f(x)\}^2}} = dx \Rightarrow \displaystyle\int \frac{df(x)}{\sqrt{1-\{f(x)\}^2}} = \int dx$

$\Rightarrow \quad \sin^{-1}(f(x)) + c = x \quad (\because f(0) = 0 \Rightarrow c = 0)$

$\therefore \quad f(x) = \sin x. \quad \therefore \quad f^{-1}(x) = \sin^{-1}x$

Then, $\displaystyle\int e^x\left(\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right)dx = e^x\sin^{-1}x + C$

26. (a): Equation of the plane can be written as

$$\frac{x}{-6} + \frac{y}{-2} + \frac{z}{3/2} = 1$$

So, the intercepts on the coordinates axes are -6, -2, $3/2$ and the required sum is $-6 - 2 + 3/2 = -13/2$

27. (c): Since roots are equal, discriminant = 0

$b^2(c-a)^2 - 4a(b-c)\,c(a-b) = 0$

$\Rightarrow \quad b^2(c-a)^2 - 4ac(ab - ac - b^2 + bc) = 0$

$\Rightarrow \quad b^2(c-a)^2 - 4a^2bc + 4a^2c^2 + 4ab^2c - 4abc^2 = 0$

$\Rightarrow \quad b^2(c-a)^2 + 4ab^2c + 4a^2c^2 - 4abc(a+c) = 0$

$\Rightarrow \quad b^2(c+a)^2 + 4a^2c^2 - 2b(a+c)\,2ac = 0$

$\Rightarrow \quad (b(c+a) - 2ac)^2 = 0 \Rightarrow b(c+a) - 2ac = 0$

$\Rightarrow \quad b = \dfrac{2ac}{a+c} \Rightarrow a, b, c$ are in H.P.

28. (a): Equation of any tangent to the parabola $P: y^2 = 8x$ is $y = mx + 2/m$

Since it touches $E: \dfrac{x^2}{4} + \dfrac{y^2}{15} = 1$

$\left(\dfrac{2}{m}\right)^2 = 4m^2 + 15 \Rightarrow m = \pm 1/2$

Equations of the tangents are $x \pm 2y + 8 = 0$

29. (a): When $m = 1/2$, the slope of the normal is -2 and equation of the normal to the parabola is

$y = -2x - 2(2)(-2) - 2(-2)^3 \Rightarrow 2x + y = 24$

30. (b): Equation of the tangent at $(2\cos\theta, \sqrt{15}\sin\theta)$ on the ellipse E to the ellipse is

$$\frac{x}{2}\cos\theta + \frac{y}{\sqrt{15}}\sin\theta = 1$$

If it represents the tangent $x - 2y + 8 = 0$, then

$\dfrac{\cos\theta}{2} = \dfrac{\sin\theta}{-2\sqrt{15}} = \dfrac{-1}{8} \Rightarrow \cos\theta = -\dfrac{1}{4}, \sin\theta = \dfrac{\sqrt{15}}{4}$

and the point of contact is $\left(\dfrac{-1}{2}, \dfrac{15}{4}\right)$.

Mock Test PAPER

West Bengal JEE 2014

The entire syllabus of Mathematics of WB-JEE is being divided into six modules, in each module there will be a Mock Test Paper (MTP) and two MTP in the whole syllabus which will be published in subsequent issues.

The syllabus for module break-up is given below :

Module I : Algebra - I, **Module - II :** Trigonometry, **Module - III :** Co-ordinate geometry of two dimensions, **Module - IV :** Differential calculus and its applications, **Module - V :** Algebra - II, **Module - VI :** Integral calculus and its applications, Differential equations.

Module – IV : DIFFERENTIAL CALCULUS AND ITS APPLICATIONS

CONTENTS :

Differential Calculus : Functions, composition of two functions and inverse of a function, limit, continuity, derivative, chain rule, derivatives of implicit functions and derivatives of functions defined parametrically.

Rolle's Theorem and Lagrange's mean value theorem (statement only), their geometric interpretation and elementary application, L'Hospital's rule (statement only) and applications, second order derivative.

Application of Calculus : Tangents and normals, conditions of tangency. Determination of monotonicity, maxima and minima. Differential coefficient as a rate of measure.

This paper is useful for JEE Main & Advanced and other Engineering Entrance Exams.

MODULE - IV : Differential Calculus and its Applications

Time : 45 mins | Marks : 30

Question numbers 1 to 24, each carries 1 mark and 25 to 27, each carries 2 marks.

SECTION-I

Q.1 to Q. 24 carries one mark each and 25 to 27 each carries 2 marks.

1. If $f(x) = (a - x^n)^{1/n}$, $a > 0$, $n \in N$, then $f(f(x)) =$

(a) 1 (b) n (c) x (d) nx

2. The function $f(x) = \dfrac{16^x - 1}{4^x}$ is

(a) even function (b) odd function

(c) periodic function (d) None of these

3. $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1 - 2x}{4x^2} =$

(a) 1/4 (b) 1/2 (c) 1 (d) –1

4. Which of the following is the domain of $\sin^{-1}\left(\log_2 \dfrac{x^2}{2}\right)$?

(a) $1 < x < 2, -2 < x < -1$ (b) $1 \le x \le 2, -2 \le x \le -1$

(c) $1 \le x < 2, -2 \le x \le 1$ (d) None of these

5. The range of the function $f(x) = \sec^{-1}[1 + \cos^2 x]$ $0 \le x \le \dfrac{\pi}{2}$ ([·] denotes the greatest integer function) is

(a) $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ (b) $\left[0, \dfrac{\pi}{2}\right]$

(c) $[0, \sec^{-1}2]$ (d) $\left\{0, \dfrac{\pi}{3}\right\}$

6. If $f(x) = \begin{cases} \dfrac{3\sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$ is continuous, then the value of k is equal to

(a) $\dfrac{3\pi}{10}$ (b) $\dfrac{3\pi}{5}$ (c) $\dfrac{\pi}{10}$ (d) $\dfrac{3\pi}{2}$

7. Let $f(x) = \begin{cases} (x-1)^2 \sin\left(\dfrac{1}{x-1}\right) - |x|, & x \neq 1 \\ -1, & x = 1 \end{cases}$ then $f(x)$ is differentiable for

(a) all x (b) all x except 0

(c) all x except 0 and 1 (d) all x except 1

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397.

8. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$ is
(a) 0 (b) –1 (c) 1 (d) 2

9. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sin^{-1}(3x - 4x^3)$ is
(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 1

10. $\frac{d}{dx}\left(\sin^2\cot^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right) =$
(a) $-\frac{1}{2}$ (b) –1 (c) 1 (d) $\frac{1}{2}$

11. The value of $\frac{dy}{dx}$, when $y = \log\left(\tan\frac{x}{2}\right) + \sin^{-1}(\cos x)$ is
(a) $\sec x + 1$ (b) $\operatorname{cosec} x + 1$
(c) $\operatorname{cosec} x - 1$ (d) $\sec x - 1$

12. If $(x + y)^7 = x^4 \cdot y^3$, then $\frac{dy}{dx} =$
(a) $\frac{x}{y}$ (b) xy (c) $\frac{y}{x}$ (d) $-\frac{x}{y}$

13. If $\lim_{x\to 0}\frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then
(a) $a = 1, b = 2, c = 1$ (b) $a = 1, b = 1, c = 2$
(c) $a = 2, b = 1, c = 1$ (d) $a = b = c = 1$

14. $\lim_{x\to\infty} x\left[\tan^{-1}\left(\frac{x+1}{x+2}\right) - \frac{\pi}{4}\right] =$
(a) 0 (b) –1 (c) $-\frac{1}{2}$ (d) $\frac{1}{2}$

15. A sphere increases its volume at the rate of π cc/sec. The rate at which its surface area increases when the radius is 1 cm is
(a) 2π sq. cm/s (b) π sq. cm/s
(c) $\frac{3\pi}{2}$ sq. cm/s (d) $\frac{\pi}{2}$ sq. cm/s

16. If $y = f\left(\frac{3x+\pi}{5x+4}\right)$ and $f'(x) = \tan^2 x$, then $\frac{dy}{dx}$ at $x = 0$ is
(a) $\frac{12+5\pi}{16}$ (b) $\frac{12-5\pi}{16}$
(c) $\frac{5+12\pi}{16}$ (d) $\frac{5-12\pi}{16}$

17. The tangent at any point P on the curve $x^{2/3} + y^{2/3} = 4$ meets the coordinate axes at A and B. Then $AB =$
(a) 2 (b) 4 (c) 8 (d) 16

18. If $f(x)$ is an even function defined on (–5, 5), then the sum of the squares of all numbers satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ is
(a) 10 (b) 12 (c) 15 (d) 8

19. The function $f(x) = \{x(x-2)\}^2$ is increasing in the set
(a) $(-\infty, 0) \cup (2, \infty)$ (b) $(-\infty, 1)$
(c) $(0, 1) \cup (2, \infty)$ (d) $(0, 2)$

20. The value of x in the interval [4, 9] at which the function $f(x) = \sqrt{x}$ satisfies the Mean Value Theorem is
(a) $\frac{13}{4}$ (b) $\frac{17}{4}$ (c) $\frac{21}{4}$ (d) $\frac{25}{4}$

21. If $y = 2\sin^{-1}\frac{x-2}{\sqrt{6}} - \sqrt{2+4x-x^2}$, then the value of $\frac{dy}{dx}$ at $x = 2$ is
(a) $-\frac{2}{\sqrt{6}}$ (b) $-\frac{1}{\sqrt{6}}$ (c) $\frac{2}{\sqrt{6}}$ (d) $\frac{1}{2}\sqrt{6}$

22. If $x\sin y + y\cos x = \pi$, then $y''(0) =$
(a) π (b) $-\pi$ (c) 1 (d) 0

23. If $y = (x + \sqrt{x^2+1})^n$, then $(1 + x^2)y_2 + xy_1 =$
(a) n^2y (b) $-n^2y$ (c) $-y$ (d) $2n^2y$

24. An open cylindrical can has to be made with 100 m^2 of tin. If its volume is maximum the ratio of its base radius and the height is
(a) 1 : 1 (b) 2 : 1 (c) 1 : 2 (d) $\sqrt{2}$: 1

25. The maximum value of $xy + 5$ subject to $2x + y = 4$ is
(a) 4 (b) 2 (c) 7 (d) 5

26. If the normal at $\left(ct, \frac{c}{t}\right)$ on the curve $xy = c^2$ meets the curve again at $\left(c\lambda, \frac{c}{\lambda}\right)$, then
(a) $\lambda = -\frac{1}{t}$ (b) $\lambda = -\frac{1}{t^3}$
(c) $\lambda^3 = -\frac{1}{t}$ (d) $\lambda^3 = \frac{1}{t}$

27. If $x = \frac{1}{z}$, $y = f(x)$ and $\frac{d^2y}{dx^2} = kz^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$, then the value of k is
(a) –1 (b) 1 (c) 2 (d) –2

SOLUTIONS

1. (c): $f(x) = (a - x^n)^{1/n}$

$\therefore\ f(f(x)) = \left[a - \{f(x)\}^n\right]^{1/n} = [a - (a - x^n)]^{1/n} = x$

2. (b): $f(x) = \frac{16^x - 1}{4^x}$ $\therefore\ f(-x) = \frac{16^{-x} - 1}{4^{-x}}$

$$= \frac{1-16^x}{4^{-x}\cdot 16^x} = -\frac{16^x-1}{4^x} = -f(x)$$

3. (b): $\lim_{x\to 0}\dfrac{e^{2x}-1-2x}{4x^2}$ $\left[\dfrac{0}{0}\right]$ form

$$= \lim_{x\to 0}\frac{2e^{2x}-2}{8x}$$

$$= \frac{1}{2}\cdot\lim_{x\to 0}\frac{e^{2x}-1}{2x} = \frac{1}{2}\left(\lim_{2x\to 0}\frac{e^{2x}-1}{2x}\right)$$

$$= \frac{1}{2}\cdot 1 \quad \left[\because \lim_{x\to 0}\frac{e^x-1}{x}=1\right]$$

$$= \frac{1}{2}$$

4. (b): $f(x) = \sin^{-1}\left(\log_2\dfrac{x^2}{2}\right)$ will be defined if

$-1 \le \log_2\dfrac{x^2}{2} \le 1$ *i.e.* $2^{-1} \le \dfrac{x^2}{2} \le 2^1$

i.e. $1 \le x^2 \le 4$

If $x^2 \le 4$, then $-2 \le x \le 2$ and if $x^2 \ge 1$, then $x \ge 1$ or $x \le -1$

$\therefore$ The domain of definition of function $f(x)$ is $1 \le x \le 2$, $-2 \le x \le -1$

5. (d): We know $0 \le \cos^2 x \le 1$

$\Rightarrow \quad 1 \le 1+\cos^2 x \le 2 \quad \therefore \quad [1+\cos^2 x] = 1, 2$

$\Rightarrow \quad \sec^{-1}[1+\cos^2 x] = \{\sec^{-1}1, \sec^{-1}2\} = \left\{0, \dfrac{\pi}{3}\right\}$

6. (a): $\lim_{x\to 0}\dfrac{3\sin \pi x}{5x} = \dfrac{3}{5}\lim_{x\to 0}\dfrac{\sin \pi x}{\pi x}\times\pi$

$$= \frac{3}{5}\pi\lim_{\pi x\to 0}\frac{\sin \pi x}{\pi x} = \frac{3}{5}\pi$$

$\because \quad f(x) = \begin{cases}\dfrac{3\sin \pi x}{5x}, & x \ne 0 \\ 2k, & x = 0\end{cases}$ is continuous

$\therefore \quad \lim_{x\to 0} f(x) = f(0) \quad \Rightarrow \quad \dfrac{3}{5}\pi = 2k$

$\therefore \quad k = \dfrac{3\pi}{10}$

7. (b): $f(x) = \begin{cases}(x-1)^2\sin\left(\dfrac{1}{x-1}\right) - |x|, & x \ne 1 \\ -1, & x = 1\end{cases}$

$\because$ $|x|$ is not differentiable at $x = 0$

Only $(x-1)^2\sin\left(\dfrac{1}{x-1}\right)$ is differentiable for all x with derivative 0 at $x = 1$.

Hence $f(x)$ is differentiable for all x except $x = 0$.

8. (b): $2^x + 2^y = 2^{x+y}$

or, $e^{x\log 2} + e^{y\log 2} = e^{(x+y)\log 2}$

or, $\log 2\left(e^{x\log 2} + e^{y\log 2}\dfrac{dy}{dx}\right) = \log 2\, e^{(x+y)\log 2}\left(1+\dfrac{dy}{dx}\right)$

or, $2^x + 2^y\dfrac{dy}{dx} = 2^{x+y}\left(1+\dfrac{dy}{dx}\right)$

or, $(2^y - 2^{x+y})\dfrac{dy}{dx} = 2^{x+y} - 2^x \quad \therefore \quad \dfrac{dy}{dx} = \dfrac{2^{x+y}-2^x}{2^y - 2^{x+y}}$

Now, $\left(\dfrac{dy}{dx}\right)_{x=y=1} = \dfrac{2^2-2}{2-2^2} = -1$

9. (a): Let $u = \sin^{-1}(2x\sqrt{1-x^2})$

$= \sin^{-1}\sin 2\theta$ (put $x = \sin\theta$)

$u = 2\theta \quad \therefore \quad \dfrac{du}{d\theta} = 2$

and $v = \sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta)$

$v = 3\theta \quad \therefore \quad \dfrac{dv}{d\theta} = 3$

Now, $\dfrac{du}{dv} = \dfrac{2}{3}$

10. (d): $\dfrac{d}{dx}\left(\sin^2\cot^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$

$$= \frac{d}{dx}\left(\sin^2\cot^{-1}\left(\tan\frac{\theta}{2}\right)\right) \qquad \text{(put } x = \cos\theta\text{)}$$

$$= \frac{d}{dx}\left(\sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) = \frac{d}{dx}\left(\cos^2\frac{\theta}{2}\right) = \frac{d}{dx}\left(\frac{1+\cos\theta}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{1+x}{2}\right) = \frac{1}{2} \qquad [\because \cos\theta = x]$$

11. (c): $y = \log\left(\tan\dfrac{x}{2}\right) + \sin^{-1}(\cos x)$

$$= \log\left(\tan\frac{x}{2}\right) + \frac{\pi}{2} - x$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\cot\frac{x}{2}\sec^2\frac{x}{2} - 1$$

$$= \frac{\cos\frac{x}{2}\cdot\sec^2\frac{x}{2}}{2\sin\frac{x}{2}} - 1 = \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} - 1$$

$$\frac{dy}{dx} = \operatorname{cosec} x - 1$$

12. (c): $(x+y)^7 = x^4\cdot y^3 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{y}{x}$

Fact : If $(x+y)^{m+n} = x^m\cdot y^n$, then $\dfrac{dy}{dx} = \dfrac{y}{x}$

13. (a): $\lim_{x\to 0}\dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$ (given)

The above limit is $\dfrac{0}{0}$ when $a - b + c = 0$ ... (i)

$\therefore$ By L.H. rule, $\lim_{x\to 0}\dfrac{ae^x + b\sin x - ce^{-x}}{x\cos x + \sin x}$

$\left[\frac{0}{0}\text{ when } a-c=0\right]$ ... (ii)

$\therefore$ By L.H. rule, $\lim\limits_{x\to 0}\dfrac{ae^x+b\cos x+ce^{-x}}{-x\sin x+\cos x+\cos x}$

$=\dfrac{a+b+c}{0+1+1}=\dfrac{a+b+c}{2}$

$\therefore\ \dfrac{a+b+c}{2}=2\ \Rightarrow\ a+b+c=4$ ... (iii)

From (i), (ii) and (iii), we get $a=1$, $b=2$, $c=1$.

14. (c): $\lim\limits_{x\to\infty} x\left[\tan^{-1}\left(\dfrac{x+1}{x+2}\right)-\dfrac{\pi}{4}\right]$

$=\lim\limits_{t\to 0}\dfrac{\tan^{-1}\dfrac{1+t}{1+2t}-\tan^{-1}1}{t}\qquad\left[\text{Put } x=\dfrac{1}{t}\right]$

$=\lim\limits_{t\to 0}\dfrac{1}{t}\tan^{-1}\left\{\dfrac{\dfrac{1+t}{1+2t}-1}{1+\dfrac{1+t}{1+2t}\cdot 1}\right\}=\lim\limits_{t\to 0}\dfrac{1}{t}\tan^{-1}\left(\dfrac{-t}{2+3t}\right)$

$=-\lim\limits_{t\to 0}\dfrac{\tan^{-1}\left(\dfrac{t}{2+3t}\right)}{\left(\dfrac{t}{2+3t}\right)}\cdot\dfrac{t}{2+3t}\cdot\dfrac{1}{t}=-1\cdot\dfrac{1}{2+3\cdot 0}=-\dfrac{1}{2}$

15. (a): $V=\dfrac{4}{3}\pi r^3\ \ \therefore\ \ \dfrac{dV}{dt}=\dfrac{4}{3}\pi(3r^2)\dfrac{dr}{dt}=4\pi r^2\dfrac{dr}{dt}$

Also, $S=4\pi r^2\ \ \therefore\ \ \dfrac{dS}{dt}=8\pi r\dfrac{dr}{dt}$

Now, $\dfrac{dS/dt}{dV/dt}=\dfrac{8\pi r}{4\pi r^2}=\dfrac{2}{r}\ \Rightarrow\ \dfrac{dS}{dt}=\dfrac{2}{r}\cdot\dfrac{dV}{dt}$

$\therefore\ \dfrac{dS}{dt}=\dfrac{2}{1}\times\pi=2\pi$ sq. cm/sec

16. (b): $y=f\left(\dfrac{3x+\pi}{5x+4}\right)$

$\dfrac{dy}{dx}=f'\left(\dfrac{3x+\pi}{5x+4}\right)\cdot\dfrac{(5x+4)3-5(3x+\pi)}{(5x+4)^2}$

$\dfrac{dy}{dx}=f'\left(\dfrac{3x+\pi}{5x+4}\right)\cdot\dfrac{12-5\pi}{(5x+4)^2}$

$\dfrac{dy}{dx}=\tan^2\left(\dfrac{3x+\pi}{5x+4}\right)\cdot\dfrac{12-5\pi}{(5x+4)^2}\quad[\because f'(x)=\tan^2 x]$

$\therefore\ \left(\dfrac{dy}{dx}\right)_{x=0}=\tan^2\dfrac{\pi}{4}\times\dfrac{12-5\pi}{(4^2)}$

$\left(\dfrac{dy}{dx}\right)_{x=0}=\dfrac{12-5\pi}{16}$

17. (c): The given equation of the curve is

$x^{2/3}+y^{2/3}=4$ ... (i)

Now, $\dfrac{2}{3}x^{-1/3}+\dfrac{2}{3}y^{-1/3}\cdot\dfrac{dy}{dx}=0$

$\therefore\ \dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}\ \Rightarrow\ \left(\dfrac{dy}{dx}\right)_{(h,k)}=-\dfrac{k^{1/3}}{h^{1/3}}$

Equation of tangent at (h, k) is

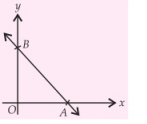
$y-k=-\dfrac{k^{1/3}}{h^{1/3}}(x-h)$

$\Rightarrow\ k^{1/3}x+h^{1/3}y=hk^{1/3}+kh^{1/3}$

$\Rightarrow\ \dfrac{x}{h^{1/3}}+\dfrac{y}{k^{1/3}}=h^{2/3}+k^{2/3}=4\qquad[\because (h,k)\text{ lies on (i)}]$

$\Rightarrow\ \dfrac{x}{4h^{1/3}}+\dfrac{y}{4k^{1/3}}=1$

$\therefore$ The tangent intersects x-axis at $A\ (4h^{1/3}, 0)$ and y-axis at $B(0, 4k^{1/3})$



$\therefore\ AB=\sqrt{4^2h^{2/3}+4^2k^{2/3}}$

$=\sqrt{4^2(h^{2/3}+k^{2/3})}$

$=\sqrt{4^2\times 4}$

$AB=4\times 2=8$ units

18. (a): $f(x)$ is an even function

$\therefore\ f(x)=f(-x)\ \Rightarrow\ f\left(\dfrac{x+1}{x+2}\right)=f(\pm x)$

$\Rightarrow\ \dfrac{x+1}{x+2}=\pm x$

$\Rightarrow\ x^2+x-1=0$ and $x^2+3x+1=0$

Now from $x^2+x-1=0$, we get $x=\dfrac{-1\pm\sqrt{5}}{2}$

and from $x^2+3x+1=0$, we get $x=\dfrac{-3\pm\sqrt{5}}{2}$

$\therefore$ Sum of squares of the above root

$2\left(\dfrac{1+5}{4}\right)+2\left(\dfrac{9+5}{4}\right)=10$

19. (c): $f(x)=(x(x-2))^2=x^2(x-2)^2=x^2(x^2-4x+4)$

or $f(x)=x^4-4x^3+4x^2$

or $f'(x)=4x^3-12x^2+8x=4x(x^2-3x+2)$

$=4x(x-1)(x-2)$

$f(x)$ will be increasing function, when $f'(x)>0$

$\Rightarrow\ 4x(x-1)(x-2)>0$

Thus the required interval $(0, 1)\cup(2, \infty)$.

20. (d): $f(x)=\sqrt{x}$

Here we find that $f(x)$ is continuous at [4, 9] and derivable in (4, 9).

$\because\ f(4)\neq f(9)$

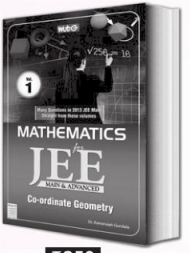
$\therefore$ By Mean Value Theorem, we get

$f'(c)=\dfrac{f(9)-f(4)}{9-4}=\dfrac{3-2}{5}=\dfrac{1}{5}$

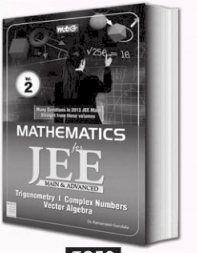
But $f'(c)=\dfrac{1}{2\sqrt{c}}$

$\therefore\ \dfrac{1}{2\sqrt{c}}=\dfrac{1}{5}\ \Rightarrow\ 4c=25\ \text{ or }\ c=\dfrac{25}{4}$

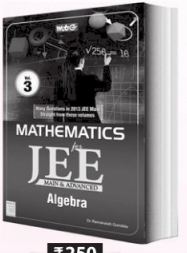
MATHEMATICS FOR JEE MAIN & ADVANCED



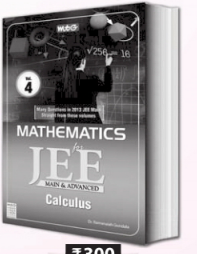
₹250



₹250



₹250



₹300

- Exclusive Theory with solved examples
- 4000+ MCQs

Buy online at www.mtg.in

For more information or to place your order:
Call at : 0124-4951200 @ Mail at : info@mtg.in
Log on to: www.mtg.in

21. (c): $y = 2\sin^{-1}\frac{x-2}{\sqrt{6}} - \sqrt{2+4x-x^2}$

$$\frac{dy}{dx} = 2\cdot\frac{1}{\sqrt{1-\left(\frac{x-2}{\sqrt{6}}\right)^2}}\times\frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{2+4x-x^2}}\times(4-2x)$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=2} = \frac{2}{\sqrt{6}}$$

22. (a): $x\sin y + y\cos x = \pi$

$$\Rightarrow \quad x\cos y\frac{dy}{dx} + \sin y + y(-\sin x) + \cos x\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x\cos y + \cos x)\frac{dy}{dx} = y\sin x - \sin y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin x - \sin y}{x\cos y + \cos x} \quad \therefore \left(\frac{dy}{dx}\right)_{x=0} = 0$$

[$\because$ $y = \pi$ when $x = 0$]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-(y\sin x - \sin y)\left(-x\sin y\frac{dy}{dx} + \cos y - \sin x\right)}{(x\cos y + \cos x)^2}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=0} = \frac{-(\pi\sin 0 - \sin\pi)(-0\cdot\sin\pi\cdot 0 + \cos\pi - 0)}{(0\cdot\cos\pi + \cos 0)^2}$$

$$= \frac{1(\pi) - (0)}{1} = \pi$$

23. (a): $y = (x + \sqrt{x^2+1})^n$

$$y_1 = n(x+\sqrt{x^2+1})^{n-1}\left\{1 + \frac{1}{2}(x^2+1)^{-1/2}\times 2x\right\}$$

$$= n(x+\sqrt{x^2+1})^{n-1}\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$\Rightarrow \quad y_1 = \frac{n(x+\sqrt{x^2+1})^n}{\sqrt{x^2+1}} \quad \Rightarrow \quad \sqrt{x^2+1}\; y_1 = ny$$

$$\Rightarrow \quad (x^2+1)y_1^2 = n^2y^2$$

$$\Rightarrow \quad (x^2+1)2y_1y_2 + y_1^2(2x) = (2yy_1)n^2$$

$$\Rightarrow \quad (x^2+1)y_2 + xy_1 = yn^2$$

24. (a): Let the radius of the base is r and height h

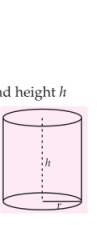
$$\therefore \quad 2\pi rh + \pi r^2 = 100$$

$$h = \frac{50}{\pi r} - \frac{r}{2}$$

$$V = \pi r^2 h = \pi r^2\left(\frac{50}{\pi r} - \frac{r}{2}\right)$$

$$= 50r - \frac{\pi r^3}{2} \quad \Rightarrow \quad \frac{dV}{dr} = 50 - \frac{3\pi r^2}{2}$$

Now, $\frac{dV}{dr} = 0$ gives $r = \frac{10}{\sqrt{3\pi}}$



$$\frac{d^2V}{dr^2} = -3\pi r \quad \therefore \left(\frac{d^2V}{dr^2}\right)_{r=\frac{10}{\sqrt{3\pi}}} < 0$$

Thus, V is maximum when $r = \frac{10}{\sqrt{3\pi}}$

$$\therefore \quad h = \frac{50}{\pi\cdot\frac{10}{\sqrt{3\pi}}} - \frac{10}{2\sqrt{3\pi}} = \frac{5\sqrt{3}}{\sqrt{\pi}} - \frac{5}{\sqrt{3\pi}} = \frac{10}{\sqrt{3\pi}}$$

So, $r : h = 1 : 1$

25. (c): Let $u = xy + 5 = x(4-2x) + 5 = 4x - 2x^2 + 5$

$$\frac{du}{dx} = 4 - 4x \quad \therefore \quad \frac{du}{dx} = 0 \text{ gives } x = 1$$

$$\frac{d^2u}{dx^2} = -4 < 0$$

$\therefore$ at $x = 1$, u is maximum

$\therefore$ $u_{max} = 4 - 2 + 5 = 7$

26. (b): The given curve $xy = c^2$ ... (i)

$$\therefore \quad y = \frac{c^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{c^2}{x^2} \quad \Rightarrow \quad \left(\frac{dx}{dy}\right)_{\left(ct, \frac{c}{t}\right)} = -\frac{c^2t^2}{c^2} = -t^2$$

Equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = t^2(x - ct) \;\Rightarrow\; yt - c = t^3x - ct^4 \quad \text{... (ii)}$$

Since (ii) is passing through $\left(c\lambda, \frac{c}{\lambda}\right)$

$$\therefore \quad \frac{ct}{\lambda} - c = t^3(c\lambda) - ct^4$$

$$\Rightarrow \quad \frac{t}{\lambda} - 1 = t^3\lambda - t^4 \Rightarrow \frac{t}{\lambda} - 1 = t^3\lambda\left(1 - \frac{t}{\lambda}\right)$$

$$\Rightarrow \quad (t^3\lambda + 1)\left(1 - \frac{t}{\lambda}\right) = 0 \quad \therefore \quad t^3\lambda = -1 \quad \text{or} \quad \lambda = -\frac{1}{t^3}$$

[$\because$ $\lambda \neq t$]

27. (c): $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \left(-\frac{1}{x^2}\right)\frac{dy}{dz}$

$$\left[\because \quad z = \frac{1}{x} \quad \therefore \quad \frac{dz}{dx} = -\frac{1}{x^2}\right]$$

Now, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{1}{x^2}\cdot\frac{dy}{dz}\right)$

$$= \frac{2}{x^3}\cdot\frac{dy}{dz} - \frac{1}{x^2}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{2}{x^3}\cdot\frac{dy}{dz} - \frac{1}{x^2}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}\cdot\frac{dy}{dz} - \frac{1}{x^2}\left(-\frac{1}{x^2}\right)\frac{d^2y}{dz^2}$$

or, $\frac{d^2y}{dx^2} = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$

$\therefore$ $k = 2$.

concept BOOSTERS

Class XI

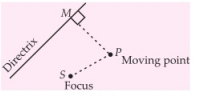
Conic Sections

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as IIT Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CONCEPTS AND FORMULAE

- A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.
  - The fixed point is called the **focus.**
  - The fixed straight line is called the **directrix.**
  - The constant ratio is called the **eccentricity** denoted by e.



$$\frac{PS}{PM} = e$$

  - The line passing through the focus & perpendicular to the directrix is called the **axis**.
  - A point of intersection of a conic with its axis is called a **vertex**.
- If S is (α, β) & directrix is $lx + my + n = 0$

then $PS = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$ and

$$PM = \frac{|lx + my + n|}{\sqrt{l^2 + m^2}}$$

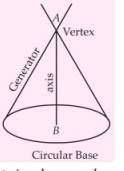
$$\frac{PS}{PM} = e \Rightarrow (l^2 + m^2)\,[(x-\alpha)^2 + (y-\beta)^2] = e^2(lx + my + n)^2$$

which is of the form

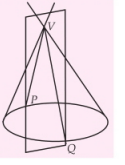
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

SECTION OF RIGHT CIRCULAR CONE BY DIFFERENT PLANES

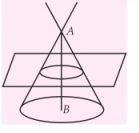
- A right circular cone is as shown in the figure.



- Section of a right circular cone by a plane passing through its vertex is a pair of straight lines passing through the vertex as shown.



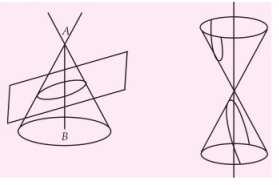
- Section of a right circular cone by a plane parallel to its base is a circle as shown in the figure.



* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

- Section of a right circular cone by a plane parallel to a generator of the cone is a parabola as shown.

- Section of a right circular cone by a plane neither parallel to any generator of the cone nor perpendicular or parallel to the axis of the cone is an ellipse or hyperbola as shown in the figure.



DISTINGUISHING VARIOUS CONICS

- The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise :

Case (I): When the focus lies on the directrix : In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and the general equation of a conic represents a pair of straight lines if:

$e > 1 \equiv h^2 > ab$, the lines will be real & distinct intersecting at S.

$e = 1 \equiv h^2 \geq ab$, the lines will coincident.

$e < 1 \equiv h^2 < ab$, the lines will be imaginary.

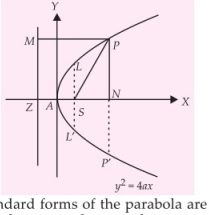
Case (II): When the focus does not lie on directrix

a parabola	an ellipse	a hyperbola	rectangular hyperbola
$e = 1$; $\Delta \neq 0$	$0 < e < 1$; $\Delta \neq 0$	$e > 1$; $\Delta \neq 0$	$e > 1$; $\Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$; $a + b = 0$

PARABOLA

DEFINITION AND TERMINOLOGY

- A parabola is the locus of a point, whose distance from a fixed point (focus) is equal to perpendicular distance from a fixed straight line (directrix).



Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

For parabola $y^2 = 4ax$:

(i) Vertex is $(0, 0)$

(ii) Focus is $(a, 0)$

(iii) Axis is $y = 0$

(iv) Directrix is $x + a = 0$

Focal distance: The distance of a point on the parabola from the focus.

Focal chord : A chord of the parabola, which passes through the focus.

Double ordinate: A chord of the parabola perpendicular to the axis of the symmetry.

Latus rectum: A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the latus rectum.

For $y^2 = 4ax \Rightarrow$ Length of the latus rectum $= 4a$.

$\Rightarrow$ ends of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$.

- **Note :**

(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus and the point of intersection of directrix and axis.

(iii) Two parabolas are said to be equal if they have the same latus rectum.

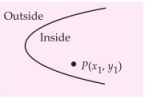
PARAMETRIC REPRESENTATION

- The simplest and the best form of representing the co-ordinates of a point on the parabola is $(at^2, 2at)$ *i.e.* the equations $x = at^2$ and $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.
- Parametric form for :

$y^2 = -4ax$	$(-at^2, 2at)$
$x^2 = 4ay$	$(2at, at^2)$
$x^2 = -4ay$	$(2at, -at^2)$

POSITION OF A POINT RELATIVE TO A PARABOLA

- The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.



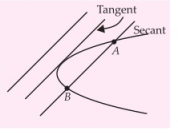
$S_1 : y_1^2 - 4ax_1$

$S_1 < 0 \rightarrow$ Inside ; $\quad S_1 > 0 \rightarrow$ Outside

LINE & A PARABOLA

- The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a \gtrless cm \Rightarrow$ condition of tangency is $c = a/m$.

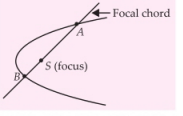
- Length of the chord intercepted by the parabola on the line $y = m x + c$ is



$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$$

Remarks

1. The equation of a chord joining t_1 and t_2 is $2x - (t_1 + t_2)\, y + 2at_1\, t_2 = 0$.
2. If t_1 and t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $t_1 t_2 = -1$. Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



3. Length of the focal chord making an angle α with the x-axis is $4a\text{cosec}^2\alpha$.

TANGENTS TO THE PARABOLA $y^2 = 4ax$

- Equation of tangent at a point on the parabola can be obtained by replacement method or using derivatives.

  In replacement method, following changes are made to the second degree equation to obtain T.

  $x^2 \rightarrow xx_1$, $y^2 \rightarrow yy_1$, $2xy \rightarrow xy_1 + x_1y$,

  $2x \rightarrow x + x_1$, $2y \rightarrow y + y_1$

  So, it follows that the tangents are :

  (i) $yy_1 = 2a(x + x_1)$ at the point (x_1, y_1) ;

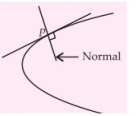
  (ii) $y = mx + \frac{a}{m} (m \neq 0)$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

  (iii) $ty = x + at^2$ at $(at^2, 2at)$.

  Note : Point of intersection of the tangents at the point t_1 and t_2 is $(at_1t_2,\ a(t_1 + t_2))$.

NORMALS TO THE PARABOLA $y^2 = 4ax$

- Normal is obtained using the slope of tangent.



Slope of tangent at $(x_1, y_1) = \frac{2a}{y_1}$

$\Rightarrow$ Slope of normal $= -\frac{y_1}{2a}$

So, equation of normal are :

(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ;

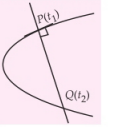
(ii) $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

Note :

(i) Point of intersection of normals at t_1 and t_2 is $((a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$



(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 and t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 and t_2 passes through a fixed point $(-2a, 0)$.

PAIR OF TANGENTS

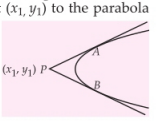
- The equation of the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by:

  $SS_1 = T^2$ where :

  $S \equiv y^2 - 4ax$;

  $S_1 = y_1^2 - 4ax_1$;

  $T \equiv yy_1 - 2a(x + x_1)$.

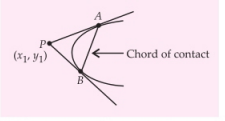


DIRECTOR CIRCLE

- Locus of the point of intersection of the perpendicular tangents to a curve is called the director circle. For parabola $y^2 = 4ax$ it's equation is $x + a = 0$ which is parabola's own directrix.

CHORD OF CONTACT

- Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$.



Note : The area of the triangle formed by the tangents from the point (x_1, y_1) and the chord of contact is $(y_1^2 - 4ax_1)^{3/2}/2a$.

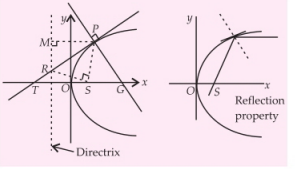
CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is

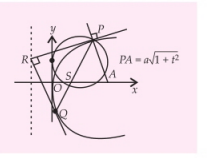
$$(y - y_1) = \frac{2a}{y_1}(x - x_1) \equiv T = S_1$$

IMPORTANT RESULTS

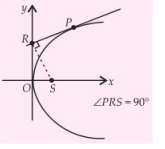
- If the tangent and normal at any point 'P' of the parabola intersect the axis at T & G then $ST = SG = SP$ where 'S' is the focus. In other words, the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP and the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



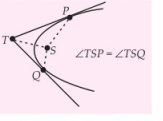
- The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.
- The tangents at the extremities of a focal chord intersect at right angles on the directrix, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point $P(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.



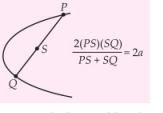
- Any tangent to a parabola and the perpendicular on it from the focus meet on the tangent at the vertex.



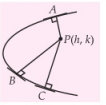
- If the tangents at P and Q meet in T, then:
  $\Rightarrow$ TP and TQ subtend equal angles at the focus S.
  $\Rightarrow$ $ST^2 = SP \cdot SQ$
  $\Rightarrow$ The triangles SPT and STQ are similar.



- Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola.



- The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.
- If normal are drawn from a point $P(h, k)$ to the parabola $y^2 = 4ax$ then $k = mh - 2am - am^3$
  i.e. $am^3 + m(2a - h) + k = 0$.
  $m_1 + m_2 + m_3 = 0$;
  $m_1m_2 + m_2m_3 + m_3m_1 = \dfrac{2a - h}{a}$; $m_1m_2m_3 = -\dfrac{k}{a}$.
  where m_1, m_2 and m_3 are the slopes of the three concurrent normals. Note that



$A, B, C \rightarrow$ Co-normal points

$\Rightarrow$ algebraic sum of the slopes of the three concurrent normals is zero.

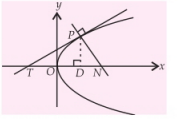
$\Rightarrow$ algebraic sum of the ordinates of the three co-normal points on the parabola is zero.

$\Rightarrow$ Centroid of the Δ formed by three co–normal points lies on the x–axis.

$\Rightarrow$ Condition for three real and distinct normals to be drawn from a point $P(h, k)$ is

$$h > 2a \quad \text{and} \quad k^2 < \frac{4}{27a}(h-2a)^3$$

- Length of subtangent at any point $P(x,y)$ on the parabola $y^2 = 4ax$ equals twice the abscissa of the point P. Note that the subtangent is bisected at the vertex.



$TD = 2(OD)$, $DN = 2a$

- Length of subnormal is constant for all points on the parabola and is equal to the semi latus rectum.

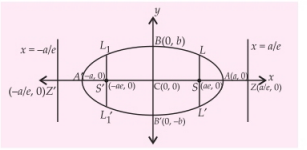
ELLIPSE

DEFINITION

- It is a locus of a point which moves in such a way that the ratio of its distance from a fixed point and a fixed line (not passes through fixed point and all points and line lies in same plane) is a constant (e) which is less than one.
The fixed point is called **focus.**
The fixed line is called **directrix.**
The constant ratio is called **eccentricity**, it is denoted by 'e'.

STANDARD EQUATION

- Standard equation of an ellipse referred to its principal axes along the co–ordinate axes is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ and $b^2 = a^2(1 - e^2)$.



- **Eccentricity:** $e = \sqrt{1 - \frac{b^2}{a^2}}, (0 < e < 1)$
- **Foci :** $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$.
- **Equations of directrices :** $x = \frac{a}{e}$ and $x = -\frac{a}{e}$
- **Major axis :** The line segment $A'A$ in which the foci S' & S lie is of length $2a$ and is called the major axis ($a > b$) of the ellipse. Point of intersection of major axis with directrix is called the foot of the directrix (Z).
- **Minor axis :** The y-axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment $B'B$ is of length $2b$ ($b < a$) is called the minor axis of the ellipse.
- **Principal axis :** The major and minor axes together are called principal axis of the ellipse.
- **Vertices :** Point of intersection of ellipse with major axis are $A' \equiv (-a, 0)$ and $A \equiv (a, 0)$.
- **Focal chord :** A chord which passes through a focus is called a focal chord.
- **Double ordinate :** A chord perpendicular to the major axis is called a double ordinate.
- **Latus rectum :** The focal chord perpendicular to the major axis is called the latus rectum.
Length of latus rectum

$$(LL') = \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$$

$= 2e$ (distance from focus to the corresponding directrix)

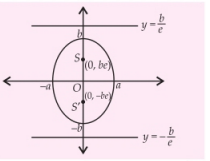
- **Centre :** The point which bisects every chord of the conic drawn through it, is called the centre of the conic. $C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Remark :

(i) If the equation of the ellipse is given as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and nothing is mentioned, then the rule is to assume that $a > b$.

(ii) If $b > a$ is given, then the y-axis will become major axis and x-axis will become the minor axis and all other points and lines will change accordingly.

Equation : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Foci : $(0, \pm be)$; Directrices : $y = \pm\frac{b}{e}$

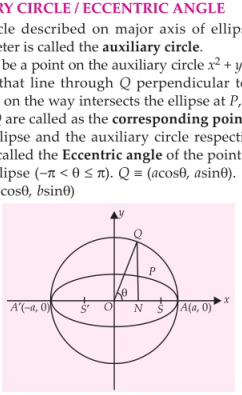
$a^2 = b^2(1 - e^2)$, $a < b \quad \Rightarrow \quad e = \sqrt{1-\frac{a^2}{b^2}}$

Vertices $(0, \pm b)$; Latus Rectum = $y = \pm be$

Length of latus rectum $= \frac{2a^2}{b}$; centre : $(0, 0)$

AUXILIARY CIRCLE / ECCENTRIC ANGLE

- A circle described on major axis of ellipse as diameter is called the **auxiliary circle**.
  Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that line through Q perpendicular to the x-axis on the way intersects the ellipse at P, then P & Q are called as the **corresponding points** on the ellipse and the auxiliary circle respectively. 'θ' is called the **Eccentric angle** of the point P on the ellipse $(-\pi < \theta \le \pi)$. $Q \equiv (a\cos\theta, a\sin\theta)$.
  $P \equiv (a\cos\theta, b\sin\theta)$



Note that : $\frac{l(PN)}{l(QN)} = \frac{b}{a} = \frac{\text{semi minor axis}}{\text{semi major axis}}$

NOTE : If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle.

PARAMETRIC REPRESENTATION

- The equations $x = a\cos\theta$ and $y = b\sin\theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where θ is a parameter. Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then; $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle. The equation of the chord of the ellipse joining two points with eccentric angles α and β is given by $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$.

POSITION OF A POINT W.R.T. AN ELLIPSE

- The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 >, <$ or $= 0$

LINE AND AN ELLIPSE

- The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is $<, =$ or $> a^2m^2 + b^2$.
  Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

TANGENTS

- **Slope form:** $y = mx \pm \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m.
- **Point form :** $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).
- **Parametric form:** $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a\cos\theta, b\sin\theta)$.
- There are two tangents to the ellipse having the same m, *i.e.* there are two tangents parallel to any given direction. These tangents touches the ellipse at extremities of a diameter.
- Point of intersection of the tangents at the point α and β is, $\left(a\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, b\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$
- The eccentric angles of the points of contact of two parallel tangents differ by π.

NORMALS

- Equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

- Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is
  $ax\sec\theta - by\,\mathrm{cosec}\,\theta = (a^2-b^2)$.
- Equation of a normal in terms of its slope 'm' is
  $$y = mx - \frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}.$$

PAIR OF TANGENTS

- The equation of the pair of tangents which can be drawn from any point (x_1, y_1) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by $SS_1 = T^2$, where
  $$S = \frac{x^2}{a^2}+\frac{y^2}{b^2}-1; \quad S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$
  $$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

DIRECTOR CIRCLE

- Locus of the point of intersection of the tangents which meet at right angles is called the **Director Circle.** The equation to this locus is $x^2+y^2=a^2+b^2$ *i.e.* a circle whose centre is the centre of the ellipse and whose radius is the length of the line joining the ends of the major and minor axes.

CHORD OF CONTACT

- Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $T = 0$, where $T = \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

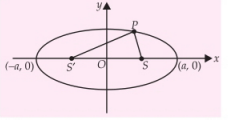
CHORD WITH A GIVEN MIDDLE POINT

- Equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose middle point is (x_1, y_1) is $T = S_1$, where
  $$S_1 = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1,\ T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

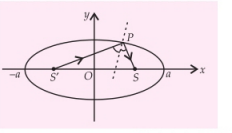
IMPORTANT RESULTS

Refering to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

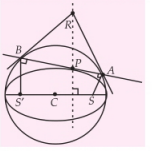
- If P be any point on the ellipse with S and S' as its foci then $l(SP) + l(S'P) = 2a$.



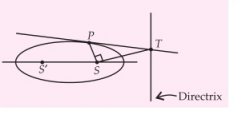
- The tangent and normal at a point P on the ellipse bisect the external and internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus and vice-versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



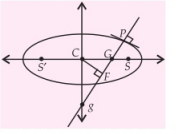
- The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.



- The portion of the tangent to an ellipse between the point of contact and the directrix subtends a right angle at the corresponding focus.

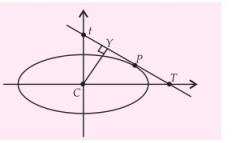


- If the normal at any point P on the ellipse with centre C meet the major and minor axes in G and g respectively and if CF be perpendicular upon this normal then
  (i) $PF \cdot PG = b^2$
  (ii) $PF \cdot Pg = a^2$
  (iii) $PG \cdot Pg = SP \cdot S'P$
  (iv) $CG \cdot C\bar{T} = CS^2$



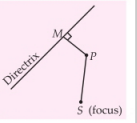
(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.
[where S and S' are the focii of the ellipse and T is the point where tangent at P meet the major axis]

- The circle on any focal distance as diameter touches the auxiliary circle. Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.
- If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre, then
(i) $Tt \cdot PY = a^2 - b^2$
(ii) least value of Tt is $a + b$.



HYPERBOLA

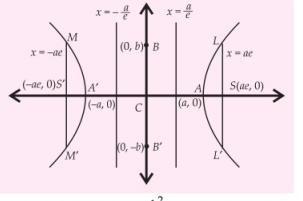
- **Definition :** A hyperbola is defined as the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point to that from a fixed line (the point does not lie on the line) is a fixed constant greater than 1.



$\frac{PS}{PM} = e > 1$, e is the eccentricity

STANDARD EQUATION & DEFINITION(S)

- Standard equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.



- **Eccentricity (e) :** $e^2 = 1 + \frac{b^2}{a^2}$
- **Foci :** $S \equiv (ae, 0)$ and $S' \equiv (-ae, 0)$.
- **Equations of directrices :**

$$x = \frac{a}{e} \quad \text{and} \quad x = -\frac{a}{e}$$

- **Transverse axis :** The line segment $A'A$ of length $2a$ in which the foci S' and S both lie is called the transverse axis of the hyperbola.
- **Conjugate axis :** The line segment $B'B$ of length $2b$ between the two points $B' \equiv (0, -b)$ and $B \equiv (0, b)$ is called as the conjugate axis of the hyperbola.
- **Principal axes :** The transverse and conjugate axis together are called principal axes of the hyperbola.
- **Vertices :** $A \equiv (a, 0)$ and $A' \equiv (-a, 0)$
- **Focal chord :** A chord which passes through the focus is called a focal chord.
- **Double ordinate :** A chord perpendicular to the transverse axis is called a double ordinate.
- **Latus rectum :** Focal chord perpendicular to the transverse axis is called latus rectum.
Its length (l) is given by

$$l = \frac{2b^2}{a} = \frac{(\text{C.A.})^2}{\text{T.A.}} = 2a(e^2 - 1)$$

(i) Length of latus rectum = $2e \times$ (distance of focus from corresponding directrix)

(ii) End points of latus rectum are $L \equiv \left(ae, \frac{b^2}{a}\right)$,

$L' \equiv \left(ae, -\frac{b^2}{a}\right)$, $M \equiv \left(-ae, \frac{b^2}{a}\right)$, $M' \equiv \left(-ae, -\frac{b^2}{a}\right)$

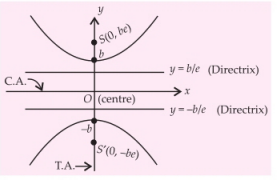
- **Centre :** The point which bisects every chord of the conic, drawn through it, is called the centre of the conic. Now, $C \equiv (0, 0)$ the origin is the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Note : Since the fundamental equation to hyperbola only differs from that to ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for hyperbola are derived from those for ellipse by simply changing the sign of b^2.

CONJUGATE HYPERBOLA

- Two hyperbolas such that transverse and conjugate axes of one hyperbola are respectively the conjugate and the transverse axes of the other are called **conjugate hyperbolas** of each other.

e.g. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are conjugate hyperbolas.



Equation : $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

Remark

(a) If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate then $e_1^{-2}+e_2^{-2}=1$.

(b) The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

(c) Two hyperbolas are said to be similar if they have the same eccentricity.

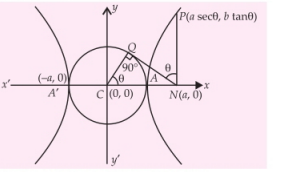
(d) Two similar hyperbolas are said to be equal if they have same latus rectum.

(e) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

- **Length of transverse axis :** The length of transverse axis = $2b$
- **Length of conjugate axis :** The length of conjugate axis = $2a$
- **Eccentricity :** $e=\sqrt{\left(1+\frac{a^2}{b^2}\right)}$
- **Foci :** The co-ordinates of the foci are $(0, \pm be)$
- **Vertices :** The co–ordinates of the vertices are $(0, \pm b)$
- **Length of latus-rectum :** The length of latus–rectum = $\frac{2a^2}{b}$
- **Equation of directrices :** The equation of directrices are $y=\pm\frac{b}{e}$

AUXILIARY CIRCLE

- A circle drawn with centre C and transverse axis as a diameter is called the **auxiliary circle** of the hyperbola. Equation of the auxiliary circle is $x^2+y^2=a^2$.
- Note from the following figure that P & Q are called the **"corresponding points"** of the hyperbola and the auxiliary circle.



PARAMETRIC REPRESENTATION

- The equations $x = a\sec\theta$ and $y = b\tan\theta$ together represent the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where θ is a parameter.
  Note that if $P(\theta) \equiv (a\sec\theta, b\tan\theta)$ is on the hyperbola then, $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.
  The equation to the chord of the hyperbola joining the two points $P(\alpha)$ and $Q(\beta)$ is given by
  $\frac{x}{a}\cos\left(\frac{\alpha-\beta}{2}\right)-\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha+\beta}{2}\right)$.

POSITION OF A POINT 'P' W.R.T. A HYPERBOLA

- The quantity $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$ is positive, zero or negative according as the point (x_1, y_1) lies outside, on or inside the curve.

LINE AND A HYPERBOLA

- The straight line $y = mx + c$ is a secant, a tangent or passes outside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as : $c^2 >, =$ or $< a^2m^2 - b^2$, respectively.

TANGENTS

- **Slope form :** $y=mx\pm\sqrt{a^2m^2-b^2}$ can be taken as the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having slope 'm'.
- **Point form :** Equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.
- **Parametric form :** Equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

Note :

(i) Point of intersection of the tangents at $P(\theta_1)$ and $Q(\theta_2)$ is

$$\left(a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},\ b\tan\left(\frac{\theta_1+\theta_2}{2}\right)\right)$$

(ii) If $|\theta_1 + \theta_2| = \pi$, then tangents at these points (θ_1 and θ_2) are parallel.

(iii) There are two parallel tangents having the same slope m. These tangents touches the hyperbola at the extremities of a diameter.

NORMALS

- The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $P(x_1, y_1)$ on it is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2=a^2e^2.$$

- The equation of the normal at the point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2=a^2e^2$$

- Equation of normal in terms of its slope 'm' are

$$y=mx\pm\frac{(a^2+b^2)m}{\sqrt{a^2-b^2m^2}}$$

PAIR OF TANGENTS

- The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $SS_1 = T^2$, where :

$$S\equiv\frac{x^2}{a^2}-\frac{y^2}{b^2}-1,\ S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1,\ T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$$

DIRECTOR CIRCLE

- The locus of the point of intersection of the tangents which are at right angles is known as the director circle of the hyperbola. The equation of the director circle is $x^2 + y^2 = a^2 - b^2$.

  If $b^2 < a^2$, then the radius of the director circle is real.

  If $b^2 = a^2$ (*i.e.* rectangular hyperbola), then the radius of the director circle is zero and it reduces to a point circle at the origin.

  If $b^2 > a^2$, then the radius of the director circle is imaginary, so that there is no such circle and so no pair of tangents at right angle can be drawn to the curve.

CHORD OF CONTACT

- Equation to the chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } T=0, \text{ where } T=\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$$

CHORD WITH A GIVEN MIDDLE POINT

- Equation of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose middle point is (x_1, y_1) is $T = S_1$,

where $S_1=\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1;\quad T\equiv\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

DIAMETER

- The locus of the middle points of a system of parallel chords with slope 'm' of hyperbola is called its diameter. It is a straight line passing through the centre of the hyperbola and has the equation

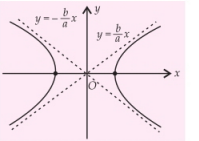
$$y=\frac{b^2}{a^2m}x.$$

NOTE : All diameters of the hyperbola pass through its centre.

ASYMPTOTES

- **Definition :** If the length of the perpendicular fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called **asymptote** of the hyperbola.

EQUATIONS OF ASYMPTOTE



$$\frac{x}{a}+\frac{y}{b}=0 \text{ and } \frac{x}{a}-\frac{y}{b}=0.$$

- **Pair of asymptotes :** $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$

Remarks :

(i) A hyperbola and its conjugate have the same asymptote.

(ii) The equation of the pair of asymptotes differs from the equation of hyperbola (or conjugate hyperbola) by the constant term only.

(iii) The asymptotes pass through the centre of the hyperbola and are equally inclined

to the transverse axis of the hyperbola. Hence the bisectors of the angles between the asymptotes are the principle axes of the hyperbola.

(iv) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(v) A simple method to find the co-ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:

Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

Remarks :

(i) No tangent to the hyperbola can be drawn from its centre.

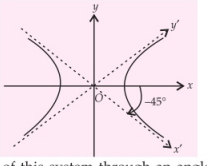
(ii) Only one tangent to the hyperbola can be drawn from a point lies on its asymptotes other than centre.

(iii) Two tangents can be drawn to the hyperbola from any of its external points which does not lie at its asymptotes.

RECTANGULAR HYPERBOLA (EQUILATERAL HYPERBOLA) :

- The particular kind of hyperbola in which the lengths of the transverse and conjugate axis are equal is called an Equilateral Hyperbola. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$. Since $a = b$ equation becomes $x^2 - y^2 = a^2$ whose asymptotes are $y = \pm x$.

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1+1} = \sqrt{2}$$



- Rotation of this system through an angle of 45° in clockwise direction gives another form to the equation of rectangular hyperbola.

which is $xy = c^2$, where $c^2 = \frac{a^2}{2}$

- **Rectangular hyperbola ($xy = c^2$) :** It is referred to its asymptotes as axes of co–ordinates.

Vertices : (c, c) and $(-c, -c)$;

Foci : $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$

Directrices : $x + y = \pm\sqrt{2}c$

Latus rectum (l) : $l = 2\sqrt{2}c = \text{T.A.} = \text{C.A}$

Parametric equation : $x = ct$, $y = c/t$, $t \in R - \{0\}$

Equation of a chord joining the points $P(t_1)$ and $Q(t_2)$ is $x + t_1t_2y = c(t_1 + t_2)$.

Equation of the tangent at $P(x_1, y_1)$ is

$$\frac{x}{x_1} + \frac{y}{y_1} = 2 \text{ and } P(t) \text{ is } \frac{x}{t} + ty = 2c$$

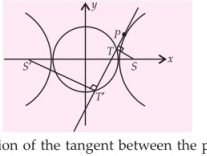
Equation of the normal at $P(t)$ is

$xt^3 - yt = c(t^4 - 1)$.

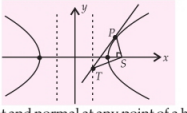
Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

IMPORTANT RESULTS

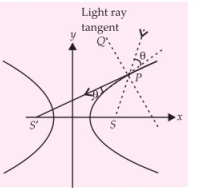
- Difference of focal distances is a constant $|PS - PS'| = 2a$
- Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle *i.e.* $x^2 + y^2 = a^2$ and the product of these perpendiculars is b^2.



- The portion of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.

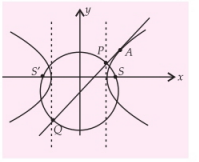


- The tangent and normal at any point of a hyperbola bisect the angle between the focal radii. This spells the reflection property of the hyperbola as **"An incoming light ray "** aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point.

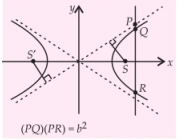


Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ $(a>k>b>0)$ are confocal and therefore orthogonal.

- The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.



- If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point and the curve is always equal to the square of the semi conjugate axis.



- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix and the common points of intersection lie on the auxiliary circle.
- The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a ΔCQR of constant area equal to ab from the asymptotes and the portion of the tangent intercepted between the asymptote is bisected at the point of contact. This implies that locus of the centre of the circle circumscribing the ΔCQR in case of a rectangular hyperbola is the hyperbola itself and for a standard hyperbola the locus would be the curve, $4(a^2x^2-b^2y^2)=(a^2+b^2)^2$.
- If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then the eccentricity of the hyperbola is $\sec\theta$.
- A rectangular hyperbola circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right), i=1,2,3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.
- If a circle and the rectangular hyperbola $xy=c^2$ meet in the four points t_1, t_2, t_3 and t_4 then
  - **(a)** $t_1t_2t_3t_4=1$
  - **(b)** the centre of the mean position of the four points bisects the distance between the centres of the two curves.
  - **(c)** the centre of the circle through the points t_1, t_2 and t_3 is $\left\{\frac{c}{2}\left(t_1+t_2+t_3+\frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+t_1+t_2+t_3\right)\right\}$

PROBLEMS

Single Correct Answer Type

1. If $x+a=0$ is the directrix of the parabola $y^2=2y+ax+2$ then the value of a is
(a) 2 (b) 3 (c) 4 (d) 4/3

2. The locus of centroid of triangle formed by a tangent to the parabola $y^2=36x$ with coordinate axes is
(a) $y^2=-9x$ (b) $y^2+3x=0$
(c) $y^2=3x$ (d) $y^2=9x$

3. If the line $y-\sqrt{3}x+3=0$ cuts the parabola $y^2=x+2$ at A and B. $P\equiv(\sqrt{3},0)$ then the value of $PA\cdot PB$ is
(a) $\frac{4}{3}(2+\sqrt{3})$ (b) $\frac{4}{3}(2-\sqrt{3})$
(c) $2\sqrt{3}$ (d) $\frac{2}{3}(2+\sqrt{3})$

4. If $a\neq 0$, and the line $2bx+3cy+4d=0$ passes through the points of intersection of the parabola $y^2=4ax$ and $x^2=4by$ then
(a) $d^2+(2b+3c)^2=0$ (b) $d^2+(3b-2c)^2=0$
(c) $d^2+(2b-3c)^2=0$ (d) $d^2+(3b+2c)^2=0$

5. The area of the closed figure bounded by $y = \frac{x^2}{2} - 2x + 2$ and the tangents to it at (1, 1/2) and (4, 2) is

(a) 9/8 sq.units (b) 3/8 sq. units
(c) 3/2 sq. units (d) 9/4 sq. units

6. If two of the three feet of normals drawn from a point to the parabola $y^2 = 4x$ be such that the slope of one tangent is double of the other then

(a) $(2, 2\sqrt{2})$ (b) (0, 0)
(c) $(2, -2\sqrt{2})$ (d) none of these

7. If the points (2, 3) and (3, 2) on a parabola are equidistant from the focus, the slope of its tangent at vertex is

(a) parallel to $y = x$ (b) parallel to $y = -xx$
(c) parallel to x-axis (d) parallel to y-axis

8. Equation of the circle of minimum radius which touches both the parabolas $y = x^2 + 2x + 4$ and $x = y^2 + 2y + 4$ is

(a) $4x^2 + 4y^2 - 11x - 11y - 31 = 0$
(b) $4x^2 + 4y^2 - 11x - 11y - 13 = 0$
(c) $4x^2 + 4y^2 - 11x - 11y + 11 = 0$
(d) $4x^2 + 4y^2 - 11x - 11y - 6 = 0$

9. Let PQ be a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which subtends an angle of $\pi/2$ radians at the centre. If L is the foot of perpendicular from (0, 0) to PQ, then

(a) locus of L is an ellipse
(b) locus of L is circle concentric with given ellipse
(c) locus of L is a hyperbola concentric with given ellipse
(d) a square concentric with given ellipse

10. From a point P, perpendicular tangents PQ and PR are drawn to ellipse $x^2 + 4y^2 = 4$. Locus of circumcentre of triangle PQR is

(a) $x^2 + y^2 = \frac{16}{5}(x^2 + 4y^2)^2$

(b) $x^2 + y^2 = \frac{5}{16}(x^2 + 4y^2)^2$

(c) $x^2 + 4y^2 = \frac{16}{5}(x^2 + y^2)^2$

(d) $x^2 + 4y^2 = \frac{5}{16}(x^2 + y^2)^2$

11. There are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose distance from the centre of the ellipse are equal to $\sqrt{\frac{a^2 + 2b^2}{2}}$. Eccentricity of this ellipse is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{\frac{2}{3}}$

12. The normal to the ellipse $4x^2 + 5y^2 = 20$ at the point 'P' touches the parabola $y^2 = 4x$. The eccentric angle of the point 'P' is

(a) $\frac{\pi}{2} + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (b) $\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$

(c) $\pi - \tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (d) $\pi - \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

13. A tangent to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$, having slope $= -\frac{4}{3}$ cuts the x and y-axis at the points A and B respectively. If O is the origin then area of triangle OAB is equal to

(a) 12 sq. units (b) 24 sq. units
(c) 36 sq. units (d) 48 sq. units

14. An ellipse with major and minor axes of length $10\sqrt{3}$ and 10 respectively, slides along the coordinate axes and always remains confined in the first quadrant. The locus of the centre of the ellipse will be the arc of a circle. The length of this arc will be equal to

(a) 10π units (b) 5π units
(c) $\frac{5\pi}{4}$ units (d) $\frac{5\pi}{3}$ units

15. The area of the triangle inscribed in an ellipse bears the ratio $\sqrt{5} : 3$ to the area of the triangle formed by joining points on the auxiliary circle corresponding to the vertices of the first triangle, then the eccentricity of the ellipse is

(a) 2/3 (b) 3/4 (c) $\sqrt{5}/3$ (d) 3/5

16. $S(3,4)$ and $S'(9,12)$ are the foci of an ellipse and the foot of perpendicular from S to a tangent to the ellipse is $(1, -4)$, then the eccentricity of the ellipse is

(a) $\frac{3}{13}$ (b) $\frac{4}{13}$ (c) $\frac{5}{13}$ (d) $\frac{7}{13}$

17. The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$ represents an ellipse if

(a) $a < 4$ (b) $a > 4$
(c) $4 < a < 10$ (d) $a > 10$

18. Chords of the parabola $y^2 = 4x$ touch the hyperbola $x^2 - y^2 = 1$. The locus of the point of intersection of the tangents drawn to the parabola at the extremities of such chords is a/an

(a) circle
(b) parabola
(c) ellipse
(d) rectangular hyperbola

19. The normal at 'P' be a hyperbola of eccentricity e, intersects its transverse and conjugate axes at L and M respectively. If locus of the mid point of LM is hyperbola, then eccentricity of the hyperbola is

(a) $\frac{e+1}{e-1}$ (b) $\frac{e}{\sqrt{e^2-1}}$

(c) e (d) $\frac{2e}{\sqrt{e^2-1}}$

20. If a ray of light incident along the line $3x + (5 - 4\sqrt{2})y = 15$ gets reflected from the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ then its reflected ray goes along the line

(a) $\sqrt{2}x - y + 5 = 0$ (b) $\sqrt{2}y - x + 5 = 0$
(c) $\sqrt{2}y - x - 5 = 0$
(d) $3x - y(4\sqrt{2} + 5) + 15 = 0$

21. If the eccentricity of the hyperbola $x^2 - y^2\sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2\sec^2\alpha + y^2 = 25$, then the value of α is

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

22. If m is the slope of a tangent to the hyperbola $\frac{x^2}{a^2 - b^2} - \frac{y^2}{a^3 - b^3} = 1$, where $a > b > 1$ then

(a) $(a+b)m^2 + ab \geq (a+b)^2$
(b) $(a+b)^2 m + ab \geq (a+b)$
(c) $abm^2 + (a+b) \geq (a+b)^2$
(d) $(a+b)m + a^2b^2 \geq (a+b)^2$

23. A normal to the hyperbola $x^2 - 4y^2 = 4$ has equal intercepts on positive x and y axes. If this normal touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then $a^2 + b^2$ is equal to

(a) 5 (b) 16 (c) 25/9 (d) 16/9

24. If a right angled triangle is inscribed in the rectangular hyperbola $xy = c^2$, then the angle between the tangent at the right angled vertex and the hypotenuse of the triangle is

(a) 0° (b) 45° (c) 60° (d) 90°

25. The equation of a hyperbola, conjugate to the hyperbola $x^2 + 3xy + 2y^2 + 2x + 3y = 0$ is

(a) $x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$
(b) $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$
(c) $x^2 + 3xy + 2y^2 + 2x + 3y + 3 = 0$
(d) $x^2 + 3xy + 2y^2 + 2x + 3y + 4 = 0$

26. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on the parabola $y^2 = 4ax$. If the circle with chord AB as a diameter touches the parabola then $|y_1 - y_2|$ is equal to

(a) $4a$ (b) $8a$
(c) $6\sqrt{2}a$ (d) not a constant

27. A rectangular hyperbola of latus rectum 4 units passes through (0, 0) and has (2, 0) as its one focus. The equation of locus of the other focus is

(a) $x^2 + y^2 = 36$ (b) $x^2 + y^2 = 4$
(c) $x^2 - y^2 = 4$ (d) $x^2 + y^2 = 9$

SECTION-II

Multiple Correct Answer Type

28. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$ then

(a) $\lambda \in (0, 1)$ (b) $\lambda \in \left(-\frac{3}{5}, 1\right)$
(c) $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$ (d) $\lambda \in \left(-\frac{3}{5}, 0\right)$

29. The equation of a parabola is $25((x-2)^2 + (y+5)^2) = (3x + 4y - 1)^2$ for this parabola

(a) tangent at vertex is $6x + 8y + 13 = 0$
(b) focus is $(2, -5)$
(c) directrix is $3x + 4y - 1 = 0$
(d) axis is $4x - 3y - 23 = 0$

30. The coordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4 is

(a) (2, 4) (b) (4, 2) (c) (2, –4) (d) (4, –2)

31. The equation of a tangent to the parabola $y^2 = 8x$ which makes an angle 45° with the line $y = 3x + 5$ is

(a) $2x + y + 1 = 0$ (b) $y = 2x + 1$
(c) $x - 2y + 8 = 0$ (d) $x + 2y + 8 = 0$

32. Extremities of the latus rectum of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ having a major axis $2a$ lies on

(a) $x^2 = a(a - y)$ (b) $x^2 = a(a + y)$
(c) $y^2 = a(a + x)$ (d) $y^2 = a(a - x)$

33. A point P moves on xy plane such that $PS + PS' = 4$ where $S(K, 0)$ and $S'(-K, 0)$ then locus of P is
(a) ellipse if $K \in (-2, 2)$
(b) pair of coincident lines if $K = \pm 2$
(c) empty if $K \in (-\infty, -2) \cup (2, \infty)$
(d) None of these

34. Let F_1, F_2 be two focii of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P. Then
(a) PN bisects $\angle F_1PF_2$ (b) PT bisects $\angle F_1PF_2$
(c) PT bisects angle $(180° - \angle F_1PF_2)$
(d) PN bisects angle $(180° - \angle F_1PF_2)$

35. If (5, 12) and (24, 7) are the focii of a conic passing through the origin then the eccentricity of conic is
(a) $\sqrt{386}/12$ (b) $\sqrt{386}/13$
(c) $\sqrt{386}/25$ (d) $\sqrt{386}/38$

36. If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which passes through $S \equiv (3, 0)$ and $PS = 2$ then length of chord PQ is
(a) 8 (b) 6 (c) 10 (d) 4

37. If pair of asymptotes of a hyperbola H is $xy - 2x - 2y = 0$. Its conjugate hyperbola H' passes through (3, 3) then
(a) eccentricity of H' is $\sqrt{2}$
(b) focus of H' is $\left(2+\frac{1}{\sqrt{2}}, 2+\frac{1}{\sqrt{2}}\right)$
(c) focus of H is $(\sqrt{2}, -\sqrt{2})$
(d) transverse axis of H is $x + y = 4$

38. Circles are drawn on chords of the rectangular hyperbola $xy = c^2$ parallel to the line $y = x$ as diameters. All such circles pass through two fixed points whose co-ordinates are
(a) (c, c) (b) $(c, -c)$
(c) $(-c, c)$ (d) $(-c, -c)$

39. The differential equation $\frac{dx}{dy} = \frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity
(a) $\sqrt{\frac{3}{5}}$ (b) $\sqrt{\frac{5}{3}}$ (c) $\sqrt{\frac{2}{5}}$ (d) $\sqrt{\frac{5}{2}}$

40. Equations of a common tangent to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is
(a) $y = x + \sqrt{a^2 - b^2}$ (b) $y = x - \sqrt{a^2 - b^2}$
(c) $y = -x + \sqrt{a^2 - b^2}$ (d) $y = -x - \sqrt{a^2 - b^2}$

SECTION-III

Comprehension Type

Paragraph for Q. No. 41 to 43

$y = f(x)$ is a parabola of the form $y = x^2 + ax + 1$. Its tangent at the point of intersection of y-axis and the parabola also touch the circle $x^2 + y^2 = r^2$. It is known that no point of the parabola is below x-axis.

41. The radius of the circle where 'a' attains its maximum value
(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) 1 (d) $\sqrt{5}$

42. The slope of the tangent when the radius of the circle is maximum
(a) 0 (b) 1
(c) –1 (d) not defined

43. The minimum area bounded by the tangent and the coordinate axes
(a) 1/4 (b) 1/3 (c) 1/2 (d) 1

Paragraph for Q. No. 44 to 46

Let the curve $C_1 : y = ax^2 + bx + c$ be such that it touches line $y = x$ at the point $x = 1$ and it passes through the point (–1, 0). Let the curve $C_2 : y = \frac{e^{-x}}{4}$ intersects the curve C_1 at point $P(m, n)$, where $m > -1$. Let
α be the length of sub-tangent of C_2 at point P.
β be the length of sub-normal of C_1 at point P.
γ be the length of sub-normal of C_2 at point P.

44. Roots of the equation $ax^2 + bx + c = 0$ are
(a) imaginary (b) real and distinct
(c) rational and equal (d) irrational

45. Angle of intersection of the curves C_1 and C_2 at the point $P(m, n)$ is
(a) $\tan^{-1}\frac{6}{7}$ (b) $\tan^{-1}\frac{7}{6}$
(c) $\tan^{-1}1$ (d) $\pi/2$

46. $\frac{\alpha}{4}, \beta, \gamma$ are in
(a) A.P. (b) G.P
(c) H.P (d) none of these

Paragraph for Q. No. 47 and 48

A rod AB of 20 units length with its ends 'A' on X-axis and B on Y-axis is sliding between the axes. P is a marked point on AB such that $AP = 8$. Locus of P is a conic.

47. The eccentricity of the conic is
(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{5}}{3}$

48. Locus of point of intersection of perpendicular tangents is

(a) $x^2 + y^2 = 200$ (b) $x^2 + y^2 = 208$

(c) $x^2 + y^2 = 313$ (d) $x^2 + y^2 = 41$

Paragraph for Q. No. 49 to 51

The normal at any point (x_1, y_1) of curve is a line perpendicular to tangent at the point (x_1, y_1). In case of parabola $y^2 = 4ax$ the equation of normal is $y = mx - 2am - am^3$ (m is slope of normal). In case of rectangular hyperbola $xy = c^2$ the equation of normal at $(ct, c/t)$ is $xt^3 - yt - ct^4 + c = 0$. The shortest distance between any two curves always exist along the common normal.

49. If normal at (5, 3) of rectangular hyperbola $xy - y - 2x - 2 = 0$ intersect it again at a point

(a) (–1, 0) (b) (–1, 1)

(c) (0, –2) (d) (3/4, –14)

50. The shortest distance between the parabola $2y^2 = 2x - 1$, $2x^2 = 2y - 1$ is

(a) $2\sqrt{2}$ (b) $\frac{1}{2\sqrt{2}}$

(c) 4 (d) $\sqrt{\frac{36}{5}}$

51. Number of normals drawn from $\left(\frac{7}{6}, 4\right)$ to parabola $y^2 = 2x - 1$ is

(a) 1 (b) 2 (c) 3 (d) 4

Paragraph for Q. No. 52 to 54

$H : x^2 - y^2 = 9$, $P : y^2 = 4(x - 5)$, $L : x = 9$.

52. If L is the chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

53. If R is the point of intersection of the tangents to H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$

(c) $y = 7$ (d) $y = 9$

54. If the chord of contact of R with respect to the parabola P meets the parabola at T and T', S is the focus of the parabola, then area of the triangle STT' is equal to

(a) 8 sq. units (b) 9 sq. units

(c) 12 sq. units (d) 16 sq. units

SECTION-IV

Matrix-Match Type

55. AB is a chord of the parabola $y^2 = 4x$ such that the normals at A and B intersect at the point $C(9, 6)$. Then match the following:

Column-I		Column-II	
(A)	The length of AB	(p)	20
(B)	The area of triangle ABC	(q)	$\frac{4}{\sqrt{13}}$
(C)	The distance from vertex of parabola to the chord AB	(r)	$\sqrt{13}$
(D)	The area bounded by the coordinate axes and the line through AB	(s)	$\frac{4}{3}$

56. Match the following:

Column-I		Column-II	
(A)	Radius of the circle passing through (3, 4) and (5, 2) having centre on $y = 2x$ is	(p)	4
(B)	Length of the common chord of $2y^2 = 3(x + 1)$, $x^2 + y^2 + 2x = 0$ is	(q)	$\sqrt{52}$
(C)	From $P(-1,2)$ tangents PA and PB are drawn to $y^2 = 4x$ then length of AB is	(r)	$\sqrt{3}$
(D)	Product of length of perpendiculars drawn from the two foci to the tangent at any point on the ellipse $25x^2 + 4y^2 = 100$ is	(s)	8

57. Match the following:

Column-I		Column-II	
(A)	From any point $P(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$ pair of tangents are drawn to the curve $x^2 + 4y^2 = 4$ which touches it at Q and R. The locus of circumcentre of ΔPQR is	(p)	straight line
(B)	The locus of the point $(3h - 2, 3k)$, where (h, k) lies on the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ is	(q)	circle
(C)	Locus of centre of circles touching two fixed circles (these two fixed circles touch each other externally and they are of different radii) is	(r)	ellipse
(D)	The extremities of a diagonal of a rectangle are (0, 0) and (4, 4). The locus of the extremities of the other diagonal is	(s)	branch of hyper-bola

58. Match the following:

Column-I		Column-II	
(A)	$P(\theta)$ is a point on 1st quadrant on ellipse $\frac{x^2}{9}+y^2=1$, $\angle POX=\phi$, then for max. value of $\theta-\phi$, θ is equal to (O is origin and OX is major axis)	(p)	$\pi/2$
(B)	If $f(x)=\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)$ then $f(2/3)=$	(q)	$\pi/4$
(C)	The angle between asymptotes of hyperbola $5x^2-2\sqrt{7}xy-y^2-2x+1=0$ is	(r)	$\pi/3$
(D)	Portion of asymptotes of the hyperbola $b^2x^2-a^2y^2=a^2b^2$ between centre and the tangent at vertex in 1st quadrant is cut by the line $y+(\tan\theta)(x-a)=0$ then θ can be	(s)	$\pi/6$

SECTION-V

Integer Answer Type

59. The line $lx+my+n=0$ touches $y^2=4x$ and $x^2=4y$ simultaneously, then $l+2m+3n=$

60. If the parabolas $y^2=\lambda x$ and $25((x-3)^2+(y+2)^2)=(3x-4y-2)^2$ are equal then $\lambda=$

61. PQ is a focal chord of $y^2=4ax$, PP' and QQ' are normal chords at P and Q respectively . Then $\frac{P'Q'}{PQ}$ is

62. The locus of point from which three normals drawn to the curve $y^2+2y-4x+5=0$ are such that two of them are perpendicular is $(y-\mu/2)^2-(x-\lambda)=0$ then the value of $\lambda-2\mu$ is equal to

63. If the equation of the curve on reflection of the ellipse $\frac{(x-4)^2}{16}+\frac{(y-3)^2}{9}=1$ about the line $x-y-2=0$ is $16x^2+9y^2+k_1x-36y+k_2=0$, then unit digit of k_1+k_2 is

64. If m is the slopes of the common tangents of the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ and the circle $x^2+y^2=3$ then m^2 is equal to

65. If m is the length of the side of square which can be made by four perpendicular tangents to the ellipse $\frac{x^2}{7}+\frac{2y^2}{11}=1$ then 'm' is equal to

66. A line is such that its intercepts on the co-ordinate axes are e and e' where e and e' are the eccentricities of a hyperbola and its conjugate hyperbola. If this line is touching a fixed circle then the area of the circle is k sq.units then [k] (where [.] g.i.f)

67. The equation of asymptotes of $xy+2x+4y+6=0$ is $xy+2x+4y+c=0$, then $c=$

SOLUTIONS

1. **(a) :** $(y-1)^2=a\left(x+\frac{3}{a}\right)$

Equation of directrix is $x+\frac{3}{a}+\frac{a}{4}=0$

$\therefore \quad a=2$

2. **(b) :** Equation of tangent to $y^2=36x$ at (x_1, y_1) is $yy_1=18(x+x_1)$ meets axis at $(-x_1,0)$, $\left(0,\frac{18x_1}{y_1}\right)$

Centroid $=(x,y)=\left(\frac{-x_1}{3},\frac{6x_1}{y_1}\right)$

$x_1=-3x; \quad y_1=\frac{6x_1}{y}$

$y_1^2=\frac{36x_1^2}{y^2} \Rightarrow 36x_1=\frac{36x_1^2}{y^2}$

$\Rightarrow \quad y^2=x_1 \Rightarrow y^2=-3x$

3. **(a) :** We have

$y=\sqrt{3}x-3$, Also $\theta=60°$ and $P(\sqrt{3},0)$

Any point on the line is $\left(\sqrt{3}+\frac{r}{2},\frac{r\sqrt{3}}{2}\right)$ ...(i)

Substituting (i) in $y^2=x+2$, we get

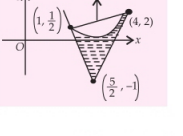
$\frac{3r^2}{4}=\sqrt{3}+\frac{r}{2}+2 \Rightarrow 3r^2=4\sqrt{3}+2r+8$

$\Rightarrow \quad 3r^2-4\sqrt{3}-2r-8=0$. So, $r_1r_2=\frac{4\sqrt{3}+8}{3}$

4. **(a) :** $y^2=4ax$, $x^2=4by$ intersect at (0, 0) and $(4a, 4a)$. Substituting these points in $2bx+3cy+4d=0$, we get $d=0$, $2b+3c=0$

So, $d^2+(2b+3c)^2=0$.

5. **(a) :**



$y = \frac{(x-2)^2}{2}$ point of intersection of tangents at $\left(1, \frac{1}{2}\right)$ and (4, 2) is $\left(\frac{5}{2}, -1\right) = (x_1, y_1)$ chord of contact is $y = \frac{x}{2}$

$$\text{Required Area} = \frac{(S_{11})^{3/2}}{2a} - \int_1^4 \left(\frac{x}{2} - \frac{(x-2)^2}{2}\right) dx$$

6. (b): Let normals be drawn from $P(h, k)$ to the parabola $y^2 = 4x$

then $k = mh - 2m - m^3$ *i.e.*, $m_1 + m_2 + m_3 = 0$

The equation of the normal at (1, 2) is $x + y = 3$ ... (1)

The equation of the normal at (1, –2) is $x - y = 3$ ... (2)

From (1) & (2) we have

$x = 3, y = 0$

$\therefore \quad h = 3, k = 0$

The equation of the third normal $y - 0 = 0(x - 3)$

$y = 0, \ x = 0$

Third feet is (0, 0).

7. (b): Equidistant points from focus are symmetric about axis of parabola

$\therefore$ tangent at vertex is parallel to the line joining the two points

$\therefore$ its slope $= \frac{3-2}{2-3} = -1$

8. (b): Given parabolas are symmetric about the line $y = x$ so they have a common normal with slope –1 it meets the parabolas at $\left(\frac{-1}{2}, \frac{13}{4}\right), \left(\frac{13}{4}, \frac{-1}{2}\right)$ hence the required circles is $x^2 + y^2 - \frac{11}{4}x - \frac{11}{4}y - \frac{13}{4} = 0$

9. (b): $PQ : x\cos\alpha + y\sin\alpha - p = 0$ ... (A)

Homogenising $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with (A)

$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ = constant.

10. (b): Let $P(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$.

Comparing chord of contact with chord with middle point $\frac{xh}{h^2 + 4k^2} + \frac{y4k}{h^2 + 4k^2} = 1$

$$\frac{x\sqrt{5}\cos\theta}{4} + \frac{y\sqrt{5}\sin\theta}{1} = 1$$

Eliminating $\theta \quad \Rightarrow \quad x^2 + y^2 = \frac{5}{16}(x^2 + 4y^2)^2$

11. (c): The given distance is clearly the length of semi major axis.

Thus, $\sqrt{\frac{a^2 + 2b^2}{2}} = a \ \Rightarrow \ 2b^2 = a^2$

$\Rightarrow \quad 2a^2(1 - e^2) = a^2 \quad \Rightarrow \ e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$

12. (d): Any normal to the ellipse is

$$\frac{\sqrt{5}x}{\cos\theta} - \frac{2y}{\sin\theta} = 1$$

i.e. $y = x.\frac{\sqrt{5}}{2}\tan\theta - \frac{\sin\theta}{2}$

it should be of the form $y = mx + \frac{1}{m}$

$$\Rightarrow \quad m = \frac{\sqrt{5}}{2}\tan\theta, \frac{1}{m} = -\frac{\sin\theta}{2}$$

$$\Rightarrow \quad \frac{\sqrt{5}}{4}.\frac{\sin^2\theta}{\cos\theta} = -1 \ \Rightarrow \ \cos\theta = -\frac{1}{\sqrt{5}}$$

$$\Rightarrow \quad \theta = \pi - \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \ \text{ or } \ \pi + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

13. (b): Equation of tangent at point $(\sqrt{18}\cos\theta, \sqrt{32}\sin\theta)$, is $\frac{x}{\sqrt{18}}\cos\theta + \frac{y}{\sqrt{32}}\sin\theta = 1$.

Its slope $= -\frac{\sqrt{32}}{\sqrt{18}}.\cot\theta = -\frac{4}{3}$ (given)

$$\Rightarrow \quad \cot\theta = 1$$

Also, $A = (\sqrt{18}\sec\theta, 0), B = (0, \sqrt{32}\text{cosec}\theta)$

$$\text{Area of } \Delta OAB = \frac{1}{2}\left|3\sqrt{2}\sec\theta \cdot 4\sqrt{2}\text{cosec}\theta\right|$$

$$= 12 \cdot \sqrt{2} \cdot \sqrt{2} = 24 \text{ sq. units}$$

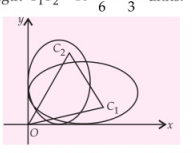
14. (d): Clearly origin will be a point lying on the director circle of the ellipse. Thus, equation of locus of centre of ellipse will be

$x^2 + y^2 = a^2 + b^2 \quad \Rightarrow \quad x^2 + y^2 = 100$

The figure shows the position of centre of ellipse in two extreme positions.

$$\theta = \angle C_1OC_2 = \frac{\pi}{2} - 2\tan^{-1}\left(\frac{5}{5\sqrt{3}}\right) = \frac{\pi}{6}$$

Thus, arc length $C_1C_2 = 10 \cdot \frac{\pi}{6} = \frac{5\pi}{3}$ units.



15. (a): $P(\theta)$, $Q(\phi)$, $R(\psi)$ be vertices of triangle inscribed in ellipses. Corresponding points on auxiliary circle $A(\theta)$, $B(\psi)$, $C(\phi)$.

$$\Delta PQR = \frac{1}{2}ab \left| \Delta PQR = \frac{1}{2}ab \right| \left| \text{Area of } \Delta ABC = \frac{1}{2}a^2 \right|$$

$$\Delta_1 : \Delta_2 = \sqrt{5} : 3 \Rightarrow \frac{b}{a} = \frac{\sqrt{5}}{3} \Rightarrow e = \frac{2}{3}.$$

16. (c) : $C = (6, 8)$, $P(1, -4)$

$CP = a = \sqrt{25 + 144}$

$SS' = 2ae = \sqrt{36 + 64} = 10 \Rightarrow ae = 5 \Rightarrow e = \frac{5}{13}$

17. (a) : For an ellipse $10 - a > 0$ and $4 - a > 0$

$\Rightarrow \quad a < 10$ and $a < 4 \Rightarrow a \in (-\infty, 4)$

18. (c) : The chord of contact $yy_0 = 2(x + x_0)$ of the point $P(x_0, y_0)$ w.r.t the parabola is tangent to the hyperbola $x^2 - y^2 = 1$ iff $2x_0^2 + y_0^2 = 4$. Locus of P is the ellipse $2x^2 + y^2 = 4$.

19. (b) : Normal $ax\cos\theta + by\cot\theta = a^2 + b^2$

$$L = \left(\frac{a^2 + b^2}{a} \sec\theta, 0 \right), M = \left(0, \frac{a^2 + b^2}{b} \tan\theta \right)$$

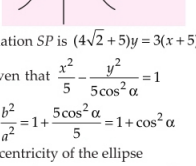
Let the mid point be $(x, y) \Rightarrow \sec\theta = \frac{2x}{ae^2}$, $\tan\theta = \frac{2by}{a^2e^2}$

$$\therefore \sec^2\theta - \tan^2\theta = 1 \Rightarrow \frac{x^2}{\frac{a^2e^4}{4}} - \frac{y^2}{\frac{a^4e^4}{4b^2}} = 1$$

$$e_1 = \sqrt{\frac{\frac{a^2e^4}{4} + \frac{a^4e^4}{4b^2}}{\frac{a^2e^4}{4}}} = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{\frac{a^2e^2}{a^2(e^2 - 1)}} = \frac{e}{\sqrt{e^2 - 1}}$$

20. (d) : $e = 5/4$

Since (5, 0) is satisfying the equation of the line $3x + (5 - 4\sqrt{2})y = 15$, so the reflected ray must pass through $(-5, 0)$ and $P = (4\sqrt{2}, 3)$



$\therefore$ The equation SP is $(4\sqrt{2} + 5)y = 3(x + 5)$

21. (b) : Given that $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

$$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha$$

Similarly, eccentricity of the ellipse

$$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1 \text{ is } e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha;$$

According to the given conditions

$$e_1 = \sqrt{3}e_2 \Rightarrow e_1^2 = 3e_2^2$$

$$\Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha \Rightarrow 2 = 4\sin^2\alpha$$

$$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4}.$$

22. (a) : $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \Rightarrow y = mx \pm \sqrt{A^2m^2 - B^2}$

where $A^2m^2 - B^2 \geq 0$

$$\Rightarrow \quad m^2 \geq \frac{B^2}{A^2} \Rightarrow m^2 \geq \frac{a^3 - b^3}{a^2 - b^2}$$

$$\Rightarrow \quad (a + b)m^2 \geq a^2 + ab + b^2 \Rightarrow (a + b)m^2 + ab \geq (a + b)^2$$

23. (c) : The equation of the normal to the hyperbola $\frac{x^4}{4} - \frac{y^2}{1} = 1$ at $(2\sec\theta, \tan\theta)$ is $2x\cos\theta + y\cot\theta = 5$.

Intercepts are equal $\Rightarrow \sin\theta = \frac{1}{2}$

$\therefore$ slope of normal $= -1$

$\therefore \quad c^2 = a^2m^2 + b^2 \Rightarrow a^2 + b^2 = 25/9$

24. (d) : Let the vertices of the ΔABC be

$$\left(ct_1, \frac{c}{t_1} \right), \left(ct_2, \frac{c}{t_2} \right) \text{ and } \left(ct_3, \frac{c}{t_3} \right)$$

Slope of AB, AC and BC are $\frac{-1}{t_1t_2}, \frac{-1}{t_1t_3}$ and $\frac{-1}{t_2t_3}$ respectively,

$$\angle ACB = 90^\circ \Rightarrow \frac{-1}{t_1t_3} \times \frac{-1}{t_2t_3} = -1 \Rightarrow t_1t_2t_3^2 + 1 = 0$$

$$\Rightarrow \quad t_1t_2 = \frac{-1}{t_3^2}.$$

$\Rightarrow$ Slope of $AB = \frac{-1}{t_1t_2} = t_3^2$ and slope of tangent at $C = \frac{-1}{t_3^2}$. Hence they intersect at 90°.

25. (b) : Equation of hyperbola + Equation of conjugate hyperbola

= (2 Equations of asymptotes) – Equation of hyperbola

$C(x, y) = 2A(x, y) - H(x, y)$ ... (i)

$\because \quad H(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y = 0$

$\therefore$ Equation of asymptotes is

$x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$

$\because \quad \Delta = 0, abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ then $\lambda = 1$

$\therefore \quad A(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$

$\therefore \quad C(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$

26. (b): Equation of circle is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

Solving with $y^2=4ax$, we get

$16a^2(y-y_1)(y-y_2)+(y^2-y_1^2)(y^2-y_2^2)=0$

$\Rightarrow (y-y_1)(y-y_2)[16a^2+(y+y_1)(y+y_2)]=0$

$\Rightarrow (y+y_1)(y+y_2)+16a^2=0$

$\Rightarrow y^2+(y_1+y_2)y+y_1y_2+16a^2=0$

The roots of the equation are equal if

$(y_1+y_2)^2=4y_2y_1+64a^2 \Rightarrow |y_1-y_2|=8a$

27. (a): The difference between the focal distances is a constant for a hyperbola. For a rectangular hyperbola latusrectum = transverse axis.

$S(2,0)$, $S'(h,k)$, $P(0,0)$,

$|S'P-SP|=4$

$\Rightarrow |\sqrt{h^2+k^2}-2|=4 \Rightarrow \sqrt{h^2+k^2}=6$

$\Rightarrow h^2+k^2=36$

Locus of (h,k) is $x^2+y^2=36$

28. (a, b, c, d): $R=\left(\frac{\lambda+1}{\lambda+1},\frac{3\lambda+1}{\lambda+1}\right)=\left(1,1+\frac{2\lambda}{\lambda+1}\right)$

$\Rightarrow y_1^2-4x_1<0 \Rightarrow \left(1+\frac{2\lambda}{\lambda+1}\right)^2-4<0$

$\Rightarrow \left(1+\frac{2\lambda}{\lambda+1}+2\right)\left(1+\frac{2\lambda}{\lambda+1}-2\right)<0$

$\Rightarrow \left(\frac{2\lambda}{\lambda+1}+3\right)\left(\frac{2\lambda}{\lambda+1}-1\right)<0$

$\Rightarrow \frac{2\lambda}{\lambda+1}>-3$ and $\frac{2\lambda}{\lambda+1}<1$

$\Rightarrow 5\lambda>-3$ and $\lambda<1 \Rightarrow \lambda>-\frac{3}{5}$ and $\lambda<1$

$\Rightarrow \lambda\in\left(-\frac{3}{5},1\right)$

29. (a, b, c, d): $\sqrt{(x-2)^2+(y+5)^2}=\left|\frac{3x+4y-1}{5}\right|$

$S(2,-5)$, Directrix $3x+4y-1=0$

$\Rightarrow$ tangent at vertex $3x+4y+K=0$ at a distance of a units from focus $(2,-5)$

Axis is $4(x-2)-3(y+5)=0$

30. (a, c): $y^2=8x$, $a=2$, $x_1+a=4$

$\therefore (x_1,y_1)=(2,\pm 4)$

31. (a, c): $m_1=3$, $\theta=45°$.

$\tan 45°=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

32. (a, b): $h=\pm ae$; $k=\pm\frac{b^2}{a}$

$k=\pm a(1-e^2)=\pm a\left(1-\frac{h^2}{a^2}\right)=\pm\left(a-\frac{h^2}{a}\right)$

On taking +ve sign, we get $k=a-\frac{h^2}{a}$

$\Rightarrow \frac{h^2}{a}=a-k \Rightarrow a-k\Rightarrow h^2=a(a-k)$

On taking –ve sign, we get

$k=-a+\frac{h^2}{a} \Rightarrow h^2=a(a+k)$.

33. (a, b, c): If PSS' form triangle then ellipse

$\therefore PS+PS'>SS' \Rightarrow K\in(-2,2)$

If PSS' are collinear the pair of coincident lines

$\Rightarrow K=\pm 2$

If PSS' neither form triangle nor collinear then empty set $\Rightarrow PS+PS'<SS'$

34. (a, c): Given that P be any point on the ellipse F_1 and F_2 are two focii of the ellipse,

$\because$ PT and PN are the tangent and normal respectively to the ellipse at point P.

By property we know that PN and PT are the internal and external bisectors of $\angle F_1PF_2$

$\therefore$ PN bisects $\angle F_1PF_2$ and PT bisects angle $(180°-\angle F_1PF_2)$

35. (a, d): For the ellipse : $a=5$ and $e=\sqrt{\frac{25-9}{25}}=\frac{4}{5}$

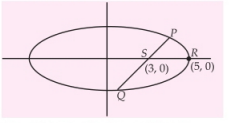
$\therefore ae=4$

$\therefore$ the foci are $(-4,0)$ and $(4,0)$

For the hyperbola, $ae=4\cdot e=2$

$\therefore a=2$ and $b^2=4(4-1)=12 \Rightarrow b=\sqrt{12}$

36. (c):



Here $SP=2$ and from the figure $SR=2$

$\therefore$ P and R are the same points

$\therefore$ length of focal chord $=2a=10$

37. (a, b, d): Let conjugate hyperbola is

$xy-2x-2y+k=0$

It passes through $(3,3)$

$\Rightarrow k=3 \Rightarrow xy-2x-2y+3=0$

$(x-2)(y-2)=1$; shifting the origin, we get equation as $xy=1$.

38. (a, d): $1/(t_1t_2)=-1$;

$(x-ct_1)(x-ct_2)+\left(y-\frac{c}{t_1}\right)\left(y-\frac{c}{t_2}\right)=1$

Use $t_1t_2=-1$ gives

$(x^2+y^2-2c^2)-(t_1+t_2)(x-y)=0 \Rightarrow S+\lambda L=0$

39. (b, d): We have, $x^2=\frac{3y^2}{2}+c$

if c is positive $\Rightarrow e=\sqrt{\frac{5}{3}}$

if c is negative $\Rightarrow \quad e = \sqrt{\frac{5}{2}}$

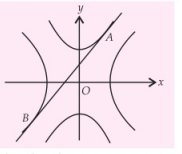
40. (a, b, c, d) : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .... (1)

and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ .... (2)

Tangent to (1) $y = mx \pm \sqrt{a^2m^2 - b^2}$

If this is also tangent to $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1$

then $a^2m^2 + b^2 = (-b^2)m^2 - (-a^2) = a^2 - b^2m^2$



$(a^2 - b^2)\, m^2 = a^2 - b^2 \Rightarrow \quad m = 1$

Hence four common tangents are $y = \pm x \pm \sqrt{a^2 - b^2}$

41. (b) **42. (a)** **43. (a)**

Point of intersection is $P(0, 1)$.

Tangent at P is $y = ax + 1$

Touches circle $\Rightarrow \quad r^2 = \frac{1}{1 + a^2}$

Point does not lie below x-axis

$\Rightarrow \quad y \geq 0 \Rightarrow \quad x^2 + ax + 1 \geq 0 \quad \forall x$

$\Rightarrow \quad a^2 - 4 \leq 0 \quad \Rightarrow \quad a \in [-2, 2]$

When a is maximum $r = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{5}}$

radius is maximum when $a = 0$

Area of triangle $= \frac{1}{2}\left|\left(\frac{-1}{a}\right)(1)\right| = \frac{1}{2a}$

Area is minimum where a is maximum.

(44-46)

Substituting the point (–1, 0) and (1, 1) in

$y = ax^2 + bx + c$ ... (A) we get

$0 = a - b + c$ ... (i) $\quad 1 = a + b + c$ ... (ii)

Differentiating (A), we get

$\frac{dy}{dx} = 2ax + b \Rightarrow \left|\frac{dy}{dx}\right|_{x=1} = 2a + b = 1$ ... (iii)

{$\because$ $y = x$ is tangent to it at $x = 1$}

(ii) – (i) $\Rightarrow \quad b = 1/2$

$a = \frac{1}{4}$ {from (iii)}

Substituting the value of a and b in (i), we get

$0 = \frac{1}{4} - \frac{1}{2} + C \Rightarrow \quad C = \frac{1}{4}$

$\therefore \quad y = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4} = \frac{(x+1)^2}{4}$

44. (c) : $0 = \frac{1}{4} - 4\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = 0 \Rightarrow$ roots are real and equal.

45. (a) : $y = \frac{(x+1)^2}{4} \Rightarrow \quad \frac{dy}{dx} = \frac{x+1}{2}$

$\left(\frac{dy}{dx}\right)_p = m_1 = \frac{1}{2}$

{as point of intersection of C_1 and C_2 is $P(0, 1/4)$}

$y = e^{-x/4} \Rightarrow \frac{dy}{dx} = -\frac{e^{-x/4}}{4}$

$\left(\frac{dy}{dx}\right)_p = m_2 = -\frac{1}{4}$

$\tan\theta = \left|\frac{\frac{1}{2} + \frac{1}{4}}{1 - \frac{1}{8}}\right| = \frac{6}{7} \Rightarrow \theta = \tan^{-1}\left(\frac{6}{7}\right)$

46. (b): $\alpha = \left|\frac{y}{dy/dx}\right| = \left|\frac{1/4}{-1/4}\right| = 1$

$\beta = \left|y.\frac{dy}{dx}\right| = \left|\frac{1}{4}.\frac{1}{2}\right| = \frac{1}{8}, \quad \gamma = \left|y.\frac{dy}{dx}\right| = \left|\frac{1}{4}\left(-\frac{1}{4}\right)\right| = \frac{1}{16}$

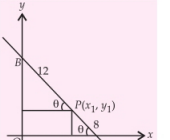
$\Rightarrow \quad \frac{\alpha}{4}, \beta, \gamma$ are in G.P.

47. (d) **48. (b)**

$x_1 = 12\cos\theta$, $y_1 = 8\sin\theta$

$\therefore \quad \frac{x_1^2}{144} + \frac{y_1^2}{64} = 1$

$\therefore$ Locus is $\frac{x^2}{144} + \frac{y^2}{64} = 1$



$\frac{4}{9} = 1 - e^2$

$e^2 = \frac{5}{9} \Rightarrow \quad e = \frac{\sqrt{5}}{3}$.

$64 = 144(1 - e^2)$

$x^2 + y^2 = 144 + 64 \quad \therefore \quad x^2 + y^2 = 208$

49. (d): $xy - y - 2x - 2 = 0$

$(x - 1)(y - 2) = 4$; $XY = 4$

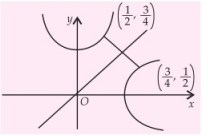
Normal at $(ct, c/t)$ intersect it again at $(ct', c/t')$ then $t' = -1/t^3$

So, $2t = 4 \Rightarrow t = 2$

$(X', Y') \equiv \left(-\frac{1}{4}, -16\right)$; $(x', y') \equiv (3/4, -14)$

50. (b): $2y\frac{dy}{dx} = 1$; $\frac{dy}{dx} = \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$

$$d = \sqrt{\frac{1}{16} + \frac{1}{16}} = \frac{1}{2\sqrt{2}}$$



51. (a): $y^2 = 2\left(x - \frac{1}{2}\right)$; $Y^2 = 2X$

For three normals $X > 1$. $x > 3/2$

$\Rightarrow$ only one normal can be drawn.

52. (b): Comparing L with $S_1 = 0$ we get the point as (1, 0).

Therefore, equation of the pair of tangents is

$(x-9)^2 = (x^2 - y^2 - 9)(-8)$ *i.e* $9x^2 - 8y^2 - 18x + 9 = 0$

53. (b): As above R = (1, 0), chord of contact of R with respect to the parabola P is

$2(x + 1) - 20 = 0 \Rightarrow x - 9 = 0$

54. (c): Substituting $x = 9$ in the equation of P, We have, $y = \pm 4$

$\therefore$ T = (9, 4) ; T' = (9, – 4) and S = (6, 0)

$\therefore$ Area of triangle STT' = 12 square units

55. A – r ; B – p ; C – q ; D – s

(A) If the parameter t_1, t_2, t_3 corresponds to the points A, B, C respectively, then

$$t_3 = 3, t_1 + t_2 = -3, t_1 t_2 = 2, t_1 - t_2 = 1 \quad \text{...(i)}$$

$$AB^2 = (t_1 - t_2)^2[(t_1 + t_2)^2 + 4] = 13 \text{ by (1)}$$

$\Rightarrow$ $AB = \sqrt{13}$

(B) Equation of AB is $2x - (t_1 + t_2)y + 2t_1t_2 = 0$

Then distance of $C(9,6)$ from this line is $\frac{40}{\sqrt{13}}$

$\therefore$ $\Delta ABC = \frac{\sqrt{13}}{2} \cdot \frac{40}{\sqrt{13}} = 20$

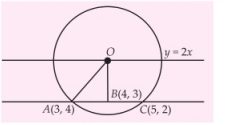
(C) Distance of (0, 0) from the line $2x + 3y + 4 = 0$ is $\frac{4}{\sqrt{13}}$

(D) The line $2x + 3y + 4 = 0$ makes intercepts of 2, 4/3 on the axes

$\therefore$ The area is 4/3

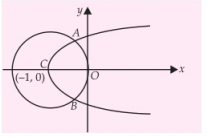
56. A – q; B – r; C – s; D – p

(A) Equation of perpendicular bisector of side AC is $x - y = 1$



So, centre O is (–1,–2); Radius = $\sqrt{52}$

(B)



Point of intersection of the two curves are

$$A\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } B\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$$

So, common chord length $AB = \sqrt{3}$.

(C) Note that $P(-1,2)$ lies on the directrix.

So, $t_1t_2 = -1$ and $l(AB) = a(t_1 - t_2)^2$

$= 1((t_1 + t_2)^2 - 4t_1t_2) = 8$

$\Rightarrow$ $t_1 + t_2 = 2$

(D) $P_1P_2 = b^2$, $\frac{x^2}{4} + \frac{y^2}{25} = 1$, so $P_1P_2 = 4$

57. A – q ; B – q ; C – s ; D – q

(A) Director circle of given ellipse is $x^2 + y^2 = 5$

and $(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$ lies on it.

$\Rightarrow$ QR is diameter of circumcircle of ΔPQR.

Let (h, k) = mid point of QR

$$T = S_1 \Rightarrow hx + 4ky = h^2 + k^2 \quad \text{... (i)}$$

$$T = 0 \Rightarrow \sqrt{5}\cos\theta x + 4\sqrt{5}\sin\theta y = 4 \quad \text{... (ii)}$$

(B) Let $\alpha = 3h - 2$, $\beta = 3k$

$\Rightarrow$ $\frac{\alpha + 2}{3} = h$ and $\frac{\beta}{3} = k$

(h, k) lies on given circle. After putting the values of h and k we will get the equation of circle.

(C) Using second definition of hyperbola.

(D) Obviously locus of the extremities of the other diagonal in the circle with given diagonal as diameter.

58. A – r ; B – r; C – r; D – (p,q,r,s)

(A) For $(\theta - \phi)$ to be maximum $3 - \tan^2\theta = 0$

$\Rightarrow$ $\theta = \pi/3$

(B) Let P and Q be point of contact of common tangent with ellipse and the circle respectively then

$P \equiv (-1, 2),\ Q \equiv (4, 2) \Rightarrow \angle POQ = \dfrac{\pi}{2}$

(C) If angle between asymptotes is θ then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \sqrt{3}$$

(D) θ lies between 0 to ∞.

59. (6) : $x + y + 1 = 0$ is common tangent

$\therefore\ l = m = n = 1.\ \ \therefore\ l + 2m + 3n = 6$

60. (6) : Parabolas are equal, if their latus rectum are equal.

Latus rectum of $y^2 = \lambda x$ is

$$\lambda(x-3)^2 + (y-(-2))^2 = \left(\frac{3x - 4y - 2}{5}\right)^2$$

$\therefore$ focus = (3, –2), directrix = $3x - 4y - 2 = 0$

$SZ = 2a = 3\dfrac{\lambda}{2} = 3 \Rightarrow \lambda = 6$

61. (3) : Let $P = (t) \Rightarrow Q = -\dfrac{1}{t}$

$\Rightarrow P' = -t - \dfrac{2}{t},\ Q' = \dfrac{1}{t} + 2t$

$$\Rightarrow P'Q' = a\left|3\left(t + \frac{1}{t}\right)\right|\sqrt{\left(t - \frac{1}{t}\right)^2 + 4}$$

$$= 3a\left(t + \frac{1}{t}\right)^2, PQ = a\left(t + \frac{1}{t}\right)^2 \Rightarrow \frac{P'Q'}{PQ} = 3$$

62. (8) : The curve is $(y + 1)^2 = 4(x - 1)$

Equation of the normal to the given curve is

$y + 1 = m(x - 1) - 2m - m^3$ which passes through (h, k)

$m^3 + m(3 - h) + 1 + k = 0$

$m_1m_2m_3 = -(1 + k) \Rightarrow m_3 = (1 + k)\ (\because\ m_1m_2 = -1)$

$(1 + k)^3 + (1 + k)(3 - h) + (1 + k) = 0$

$(1 + k)^2 + 3 - h + 1 = 0 \Rightarrow (y + 1)^2 = (x - 4)$

$\therefore\ \dfrac{\mu}{2} = -1 \Rightarrow \mu = -2,\ \lambda = 4\ \ \therefore\ \lambda - 2\mu = 8$

63. (2) : The reflection of α, β in the line $ax + by + c = 0$ is given as $\dfrac{x - \alpha}{a} = \dfrac{y - \beta}{b} = -\dfrac{2 \times (\alpha a + b\beta + c)}{a^2 + b^2}$

We have $\dfrac{(x-4)^2}{16} + \dfrac{(y-3)^2}{9} = 1$ any point P on the curve is $4(1 + \cos\theta),\ 3(1 + \sin\theta)$

The reflection of the point P about the line $x - y - 2 = 0$ can be found using

$$\frac{h - 4(1 + \cos\theta)}{1} = \frac{k - 3(1 + \sin\theta)}{-1}$$

$$= \frac{-2[4(1 + \cos\theta) - 3(1 + \sin\theta) - 2]}{2}$$

$= -(4\cos\theta - 3\sin\theta) + 1$

$\therefore\ h = 4(1 + \cos\theta) - 4\cos\theta + 3\sin\theta + 1 = 5 + 3\sin\theta$

and $k = 3(1 + \sin\theta) + 4\cos\theta - 3\sin\theta - 1 = 2 + 4\cos\theta$

$$\left(\frac{h-5}{3}\right)^2 + \left(\frac{k-2}{4}\right)^2 = 1$$

$\therefore$ Equation of image is

$16(x - 5)^2 + 9(y - 2)^2 = 144$

$\Rightarrow 16x^2 + 9y^2 - 160x - 36y + 292 = 0$

On comparing with given equation, we get

$k_1 = -160,\ k_2 = 292$

$\therefore\ k_1 + k_2 = 132 \Rightarrow$ unit's digit = 2

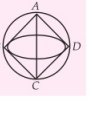
64. (2) : Let m be the slope of the common tangent, then $\pm\sqrt{3}\sqrt{1 + m^2} = \pm\sqrt{4m^2 + 1}$

$\Rightarrow\ 3 + 3m^2 = 4m^2 + 1 \Rightarrow m^2 = 2 \Rightarrow m = \pm\sqrt{2}$

65. (5) : (i) Clearly, vertices of the square lie on the director circle of the ellipse $\dfrac{x^2}{7} + \dfrac{2y^2}{11} = 1$

which is $x^2 + y^2 = 7 + \dfrac{11}{2}$ or $x^2 + y^2 = \dfrac{25}{2}$

(ii) Clearly, $AC = 2\sqrt{\dfrac{25}{2}}$



Now $AB = BC = CD = AD$ and in ΔACD, $AC^2 = CD^2 + AD^2$

$$\Rightarrow\ 2AD^2 = \left(2\sqrt{\frac{25}{2}}\right)^2$$

$\Rightarrow\ AD^2 = 25 \Rightarrow\ AD = 5$ units

66. (3) : $\dfrac{1}{e^2} + \dfrac{1}{e'^2} = 1;\ \dfrac{x}{e} + \dfrac{y}{e'} = 1$

$$\Rightarrow \left|\frac{\frac{0}{e} + \frac{0}{e'} - 1}{\sqrt{\frac{1}{e^2} + \frac{1}{e'^2}}}\right| = 1$$

$\Rightarrow$ Perpendicular distance from (0, 0) to $\dfrac{x}{e} + \dfrac{y}{e'} = 1$ is '1'

$\therefore$ A circle with centre at (0, 0) and radius '1' unit always touches the line $\dfrac{x}{e} + \dfrac{y}{e'} = 1$.

67. (8) : $xy + 2x + 4y + C = 0$ represents pair of lines

$\Rightarrow\ C = 8$

■■

concept BOOSTERS

Class XII

Trigonometry II

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as IIT, AIEEE, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

CONCEPTS AND FORMULAE

- **SINE RULE**

In any triangle ABC, the sines of the angles are proportional to the opposite sides

$$i.e.\ \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

- **COSINE FORMULA**

In any ΔABC

(i) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

or, $a^2 = b^2 + c^2 - 2bc\cos A$
$= b^2 + c^2 + 2bc\cos(B + C)$

(ii) $\cos B=\dfrac{c^2+a^2-b^2}{2ca}$

(iii) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}$

- **PROJECTION FORMULA**

In any ΔABC

(i) $a = b\cos C + c\cos B$
(ii) $b = c\cos A + a\cos C$
(iii) $c = a\cos B + b\cos A$

- **NAPIER'S ANALOGY - TANGENT RULE**

In any ΔABC

(i) $\tan\dfrac{B-C}{2}=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(ii) $\tan\dfrac{C-A}{2}=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(iii) $\tan\dfrac{A-B}{2}=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

- **TRIGONOMETRIC FUNCTIONS OF HALF ANGLES**

(i) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$; $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{ca}}$;

$\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

(ii) $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$; $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$;

$\cos\dfrac{C}{2}=\sqrt{\dfrac{s(s-c)}{ab}}$

(iii) $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}=\dfrac{\Delta}{s(s-a)}=\dfrac{(s-b)(s-c)}{\Delta}$

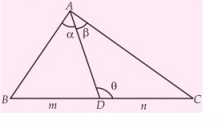
where $s=\dfrac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.

(iv) $\sin A=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\dfrac{2\Delta}{bc}$

- **AREA OF TRIANGLE (Δ)**

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B$$
$$=\sqrt{s(s-a)(s-b)(s-c)}$$

- ***m-n* RULE**



* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$

- **RADIUS OF CIRCUMCIRCLE**

If R be the circumradius of ΔABC, then

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

- **RADIUS OF THE INCIRCLE**

If 'r' be the inradius of ΔABC, then

(i) $r = \dfrac{\Delta}{s}$

(ii) $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

(iii) $r = \dfrac{a\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}}$ and so on

(iv) $r = 4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

- **RADIUS OF THE EX- CIRCLES**

If r_1, r_2, r_3 are the radii of the ex-circles of ΔABC opposite to the vertex A, B, C respectively, then

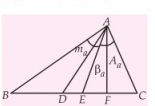
(i) $r_1 = \dfrac{\Delta}{s-a}; r_2 = \dfrac{\Delta}{s-b}; r_3 = \dfrac{\Delta}{s-c}$

(ii) $r_1 = s\tan\dfrac{A}{2}; r_2 = s\tan\dfrac{B}{2}; r_3 = s\tan\dfrac{C}{2}$

(iii) $r_1 = \dfrac{a\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}}$ and so on

(iv) $r_1 = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

- **LENGTH OF ANGLE BISECTORS, MEDIANS & ALTITUDES**



(i) Length of an angle bisector from the angle A

$= \beta_a = \dfrac{2bc\cos\dfrac{A}{2}}{b+c}$

(ii) Length of median from the angle $A = m_a$

$= \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

(iii) Length of altitude from the angle $A = A_a$

$= \dfrac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

- **THE DISTANCES OF THE SPECIAL POINTS FROM VERTICES AND SIDES OF TRIANGLE**

(i) Circumcentre (O) : $OA = R$ and $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\dfrac{A}{2}$ and $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\text{cosec}\dfrac{A}{2}$ and $I_{1a} = r_1$

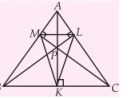
(iv) Orthocentre (H) : $HA = 2R\cos A$ and $Ha = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \dfrac{1}{3}\sqrt{2b^2 + 2c^2 - a^2}$

and $G_a = \dfrac{2\Delta}{3a}$

- **ORTHOCENTRE AND PEDAL TRIANGLE**

The triangle KLM which is formed by joining the feet of the altitudes is called the Pedal Triangle.



(i) Its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$

(ii) Its sides are $a\cos A = R\sin 2A$, $b\cos B = R\sin 2B$ and $c\cos C = R\sin 2C$

(iii) Circumradii of the triangles PBC, PCA, PAB and ABC are equal.

- **EXCENTRAL TRIANGLE**

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

(ii) Its angles are $\dfrac{\pi}{2} - \dfrac{A}{2}, \dfrac{\pi}{2} - \dfrac{B}{2}$ and $\dfrac{\pi}{2} - \dfrac{C}{2}$

(iii) Its sides are $4R\cos\dfrac{A}{2}, 4R\cos\dfrac{B}{2}, 4R\cos\dfrac{C}{2}$

(iv) $II_1 = 4R\sin\dfrac{A}{2}$, $II_2 = 4R\sin\dfrac{B}{2}$, $II_3 = 4R\sin\dfrac{C}{2}$

(v) Incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

- **DISTANCE BETWEEN SPECIAL POINTS**

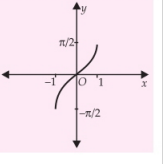
(i) Distance between circumcentre and orthocentre $OH^2 = R^2(1 - 8\cos A\cos B\cos C)$

(ii) Distance between circumcentre and incentre $OI^2 = R^2\left(1 - 8\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}\right) = R^2 - 2Rr$

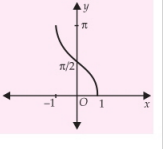
(iii) Distance between circumcentre and centroid

$$OG^2 = R^2 - \frac{1}{9}(a^2 + b^2 + c^2)$$

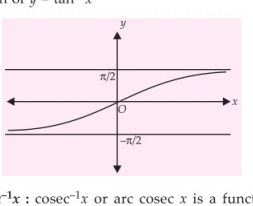
- **$\sin^{-1}x$:** The symbol $\sin^{-1}x$ or $\arcsin x$ denote the angle θ so that $\sin\theta = x$. As a direct meaning, $\sin^{-1}x$ is not a function, as it does not satisfy the requirements for a rule to become a function. But by a suitable choice $[-1, 1]$ as its domain and standardized set $[-\pi/2, \pi/2]$ as its range, then rule $\sin^{-1}x$ is a single valued function.
  Thus $\sin^{-1}x$ is considered as a function with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.
  The graph of $y = \sin^{-1}x$ is as shown below, which is obtained by taking the mirror image, of the portion of the graph of $y = \sin x$, from $x = -\pi/2$ to $x = \pi/2$, on the line $y = x$.



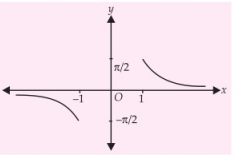
- **$\cos^{-1}x$:** We have $\cos^{-1}x$ or arc cos x as a function with domain $[-1, 1]$ and range $[0, \pi]$.
  The graph of $y = \cos^{-1}x$ is similarly obtained as the mirror image of the portion of the graph of $y = \cos x$ from $x = 0$ to $x = \pi$.



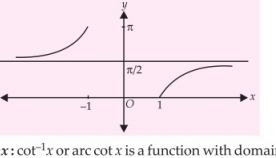
- **$\tan^{-1}x$:** We get $\tan^{-1}x$ or arc $\tan x$ as a function with domain R and range $(-\pi/2, \pi/2)$.
  Graph of $y = \tan^{-1}x$



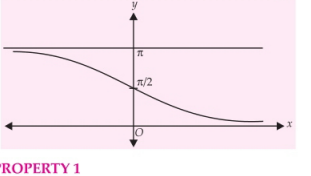
- **$\operatorname{cosec}^{-1}x$:** $\operatorname{cosec}^{-1}x$ or arc cosec x is a function with domain $R - (-1, 1)$ and range $[-\pi/2, \pi/2] - \{0\}$.
  Graph of $y = \operatorname{cosec}^{-1}x$



- **$\sec^{-1}x$:** $\sec^{-1}x$ or arc sec x is a function with domain $R - (-1, 1)$ and range $[0, \pi] - \{\pi/2\}$.
  Graph of $y = \sec^{-1}x$



- **$\cot^{-1}x$:** $\cot^{-1}x$ or arc cot x is a function with domain R and range $(0, \pi)$. Graph of $y = \cot^{-1}x$



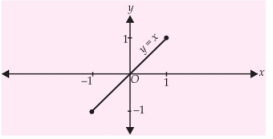
- **PROPERTY 1**
  (i) The graphs of $\sin^{-1}x$, $\tan^{-1}x$, $\operatorname{cosec}^{-1}x$ are symmetric about origin.
  Hence we get $\sin^{-1}(-x) = -\sin^{-1}x$
  $\tan^{-1}(-x) = -\tan^{-1}x$
  $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$.
  (ii) Also the graphs of $\cos^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$ are symmetric about the point $(0, \pi/2)$. From this, we get $\cos^{-1}(-x) = \pi - \cos^{-1}x$
  $\sec^{-1}(-x) = \pi - \sec^{-1}x$
  $\cot^{-1}(-x) = \pi - \cot^{-1}x$.

- **PROPERTY 2**
  (i) $\sin(\sin^{-1}x) = x$, $-1 \le x \le 1$
  Proof : Let $\theta = \sin^{-1}x$. Then $x \in [-1, 1]$ & $\theta \in [-\pi/2, \pi/2]$.
  $\Rightarrow \sin\theta = x$, by meaning of the symbol
  $\Rightarrow \sin(\sin^{-1}x) = x$
  Similar proofs can be carried out to obtain
  (ii) $\cos(\cos^{-1}x) = x$, $-1 \le x \le 1$
  (iii) $\tan(\tan^{-1}x) = x$, $x \in R$
  (iv) $\cot(\cot^{-1}x) = x$, $x \in R$

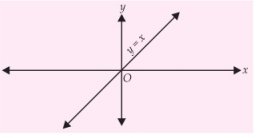
(v) $\sec(\sec^{-1}x) = x,\ x \le -1,\ x \ge 1$

(vi) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x,\ |x| \ge 1$

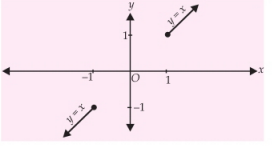
The graph of $y = \sin(\sin^{-1}x) \equiv \cos(\cos^{-1}x)$



The graph of $y = \tan(\tan^{-1}x) \equiv \cot(\cot^{-1}x)$



The graph of $y = \operatorname{cosec}(\operatorname{cosec}^{-1}x) \equiv \sec(\sec^{-1}x)$



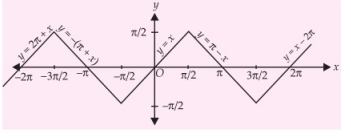
- **PROPERTY 3**

(i) $\sin^{-1}(\sin x)$

$$= \begin{cases} -2n\pi + x, & x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right] \\ (2n+1)\pi - x, & x \in \left[(2n+1)\pi - \frac{\pi}{2}, (2n+1)\pi + \frac{\pi}{2}\right] \end{cases}, \quad n \in z$$

Proof : If $x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2]$, then $-2n\pi + x \in [-\pi/2,\ \pi/2]$ and $\sin(-2n\pi + x) = \sin x$. Hence $\sin^{-1}(\sin x) = -2n\pi + x$ for $x \in [2n\pi - \pi/2,\ 2n\pi + \pi/2]$.

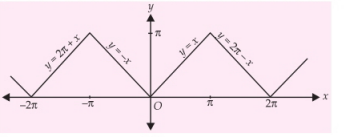
Graph of $y = \sin^{-1}(\sin x)$



(ii) $\cos^{-1}(\cos x)$

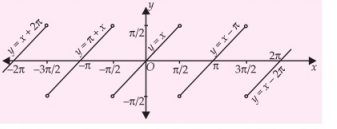
$$= \begin{cases} -2n\pi + x, & x \in [2n\pi, (2n+1)\pi] \\ 2n\pi - x, & x \in [(2n-1)\pi, 2n\pi] \end{cases}, n \in I$$

Graph of $y = \cos^{-1}(\cos x)$



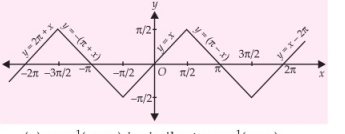
(iii) $\tan^{-1}(\tan x) = -n\pi + x,\ n\pi - \pi/2 < x < n\pi + \pi/2,\ n \in Z$

Graph of $y = \tan^{-1}(\tan x)$



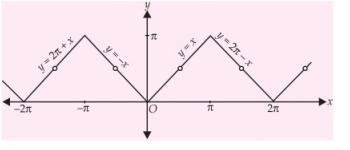
(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ is similar to $\sin^{-1}(\sin x)$

Graph of $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$



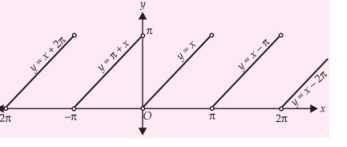
(v) $\sec^{-1}(\sec x)$ is similar to $\cos^{-1}(\cos x)$

Graph of $y = \sec^{-1}(\sec x)$



(vi) $\cot^{-1}(\cot x) = -n\pi + x,\ x \in (n\pi,\ (n+1)\pi),\ n \in Z$

Graph of $y = \cot^{-1}(\cot x)$



Remark : $\sin(\sin^{-1}x)$, $\cos(\cos^{-1}x)$, ..., $\cot(\cot^{-1}x)$ are aperiodic (non periodic) functions where as $\sin^{-1}(\sin x)$, ..., $\cot^{-1}(\cot\, x)$ are periodic functions.

- **PROPERTY 4**

(i) $\text{cosec}^{-1}(x) = \sin^{-1}(1/x),\ |x| \geq 1$

Proof : Let $\text{cosec}^{-1}x = \theta \Rightarrow 1/x = \sin\theta$

$\Rightarrow \sin^{-1}(1/x) = \sin^{-1}(\sin\theta)$

$= \theta (\text{as } \theta \in [-\pi/2, \pi/2]) = \text{cosec}^{-1}x$

(ii) $\sec^{-1}x = \cos^{-1}\{1/x\},\ |x| \geq 1$

(iii) $\cot^{-1}x = \begin{cases} \tan^{-1}(1/x), & x > 0 \\ \pi + \tan^{-1}(1/x), & x < 0 \end{cases}$

- **PROPERTY 5**

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2},\ -1 \leq x \leq 1$

Proof : Let $A = \sin^{-1}x$ and $B = \cos^{-1}x$

$\Rightarrow \sin A = x$ and $\cos B = x$

$\Rightarrow \sin A = \cos B \Rightarrow \sin A = \sin(\pi/2 - B)$

$\Rightarrow A = \pi/2 - B$, because A and $\pi/2 - B \in [-\pi/2, \pi/2] \Rightarrow A + B = \pi/2$.

Similarly, we can prove

(ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},\ x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \dfrac{\pi}{2},\ |x| \geq 1$

- **PROPERTY 6 : IDENTITIES ON ADDITION AND SUBTRACTION**

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$;

$x \geq 0,\ y \geq 0$ and $(x^2 + y^2) \leq 1$

$= \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$,

$x \geq 0,\ y \geq 0$ and $x^2 + y^2 \geq 1$

Proof : Let $A = \sin^{-1}x$ and $B = \sin^{-1}y$, where $x, y \in [0, 1]$.

$\sin(A + B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \sin^{-1}(\sin(A+B)) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$\Rightarrow \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$= \begin{cases} A + B, & \text{for } 0 \leq A + B \leq \pi/2 \\ \pi - (A + B), & \text{for } \pi/2 \leq A + B \leq \pi \end{cases}$

$= \begin{cases} \sin^{-1}x + \sin^{-1}y, & x^2 + y^2 \leq 1 \\ \pi - (\sin^{-1}x + \sin^{-1}y), & x^2 + y^2 \geq 1 \end{cases}$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$;

$x, y \in [0, 1]$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$;

$x, y \in [0, 1]$

(iv) $\cos^{-1}x - \cos^{-1}y$

$= \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}); \\ \qquad 0 \leq x < y \leq 1 \\ -\cos^{-1}(xy + \sqrt{1-x^2}\sqrt{1-y^2}); \\ \qquad 0 \leq y < x \leq 1 \end{cases}$

(v) $\tan^{-1}x + \tan^{-1}y$

$= \begin{cases} \pi/2, & \text{if } x, y > 0 \text{ \& } xy = 1 \\ -\pi/2, & \text{if } x, y < 0 \text{ \& } xy = 1 \\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x, y \geq 0 \text{ \& } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x, y \geq 0 \text{ \& } xy > 1 \end{cases}$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right),\ x \geq 0,\ y \geq 0$

Remarks :

(i) $x^2 + y^2 \leq 1$ & $x, y \geq 0$

$\Rightarrow 0 \leq \sin^{-1}x + \sin^{-1}y \leq \dfrac{\pi}{2}$

and $x^2 + y^2 \geq 1$ & $x, y \geq 0$

$\Rightarrow \dfrac{\pi}{2} \leq \sin^{-1}x + \sin^{-1}y \leq \pi$

(ii) $xy < 1$ and $x, y \geq 0$

$\Rightarrow 0 \leq \tan^{-1}x + \tan^{-1}y < \dfrac{\pi}{2}$; $xy > 1$ and

$x, y \geq 0 \Rightarrow \dfrac{\pi}{2} < \tan^{-1}x + \tan^{-1}y < \pi$

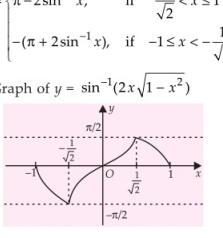
(iii) For $x < 0$ or $y < 0$ these identities can be used with the help of property "$-x$" *i.e.* change x or y to $-x$ or $-y$ which are positive.

- **PROPERTY 7 : MORE USEFUL IDENTITIES**

(i) $\sin^{-1}(2x\sqrt{1-x^2})$

$= \begin{cases} 2\sin^{-1}x, & \text{if } |x| \leq \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x, & \text{if } \dfrac{1}{\sqrt{2}} < x \leq 1 \\ -(\pi + 2\sin^{-1}x), & \text{if } -1 \leq x < -\dfrac{1}{\sqrt{2}} \end{cases}$

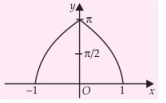
Graph of $y = \sin^{-1}(2x\sqrt{1-x^2})$



(ii) $\cos^{-1}(2x^2-1)$

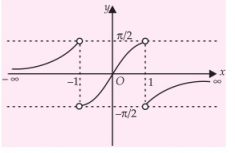
$$= \begin{cases} 2\cos^{-1}x, & \text{if} \quad 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x, & \text{if} \quad -1 \le x < 0 \end{cases}$$

Graph of $y = \cos^{-1}(2x^2 - 1)$



(iii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x, & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x), & \text{if } x > 1 \end{cases}$

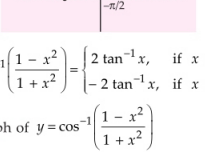
Graph of $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$



(iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

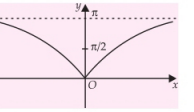
$$= \begin{cases} 2\tan^{-1}x, & \text{if} \quad |x| \le 1 \\ \pi - 2\tan^{-1}x, & \text{if} \quad x > 1 \\ -(\pi + 2\tan^{-1}x), & \text{if} \quad x < -1 \end{cases}$$

Graph of $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$



(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & \text{if} \quad x \ge 0 \\ -2\tan^{-1}x, & \text{if} \quad x < 0 \end{cases}$

Graph of $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$



(vi) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then $x + y + z = xyz$

(vii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$, then $xy + yz + zx = 1$

(viii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$

(ix) $\tan^{-1}1 + \tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \dfrac{\pi}{2}$

SALIENT POINTS

- $\tan\dfrac{A}{2}\cdot\tan\dfrac{B}{2} = \dfrac{s-c}{s}$, $\cot\dfrac{A}{2}\cdot\cot\dfrac{B}{2} = \dfrac{s}{s-c}$,
- $\tan\dfrac{A}{2} + \tan\dfrac{B}{2} = \dfrac{c}{s}\cot\dfrac{C}{2}$
- $\tan\dfrac{A}{2} - \tan\dfrac{B}{2} = \dfrac{a-b}{\Delta}(s-c)$
- $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} = \dfrac{\tan\dfrac{A}{2} + \tan\dfrac{B}{2}}{\tan\dfrac{A}{2}.\tan\dfrac{B}{2}} = \dfrac{c}{s-c}\cot\dfrac{C}{2}$
- $\sum(p-q) = (p-q) + (q-r) + (r-p) = 0$ …(i)
- $\sum p(q-r) = p(q-r) + q(r-p) + r(p-q) = 0$ …(ii)
- $\sum(p+a)(q-r) = \sum p(q-r) + a\sum(q-r) = 0$ by (i) and (ii)
- If A, B, C be the angles of a triangle, then

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

(ii) $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(iii) $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

(iv) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

- If I_n represents area of n-sided regular polygon inscribed in a unit circle. O_n be the area of the n-sided regular polygon circumscribing it, then

$$I_n = \frac{O_n}{2}\left[1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}\right]$$

(The given circle is a unit circle

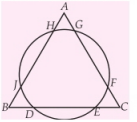
$$\therefore \quad I_n = \frac{n}{2}\sin\frac{2\pi}{n},\ O_n = n\tan\frac{\pi}{n}\Big)$$

- Ratio in which incentre '*I*' divides angle bisector *AD* of ΔABC with *D* on *BC* is given by $\frac{AI}{DI}=\frac{b+c}{a}$.
- Ratio in which orthocentre '*H*' divides altitude *AD* of ΔABC with *D* on *BC* is given by $\frac{AH}{DH}=\frac{\cos A}{\cos B\cos C}$.
- In a triangle, $R \geq 2r$.

PROBLEMS

Single Correct Answer Type

1. In the given figure, the circle meets the sides of an equilateral triangle at six points. If $AG = 2$, $GF = 13$, $FC = 1$ and $HJ = 7$, then $DE =$



(a) $2\sqrt{22}$

(b) $7\sqrt{3}$

(c) 9

(d) 10

2. If $\tan^2\left(\frac{\pi-A}{4}\right)+\tan^2\left(\frac{\pi-B}{4}\right)+\tan^2\left(\frac{\pi-C}{4}\right)=1$, then ΔABC is

(a) equilateral (b) isosceles

(c) scalene (d) none of these

3. If *P* is a point on the altitude *AD* of the triangle *ABC* such that $\angle CBP=\frac{B}{3}$ then *AP* is equal to

(a) $2a\sin\frac{C}{3}$ (b) $2b\sin\frac{A}{3}$

(c) $2c\sin\frac{B}{3}$ (d) $2c\sin\frac{C}{3}$

4. In triangle *ABC*, $a^2 + c^2 = 2002b^2$ then $\frac{\cot A+\cot C}{\cot B}=$

(a) $\frac{1}{2001}$ (b) $\frac{2}{2001}$

(c) $\frac{3}{2001}$ (d) $\frac{4}{2001}$

5. If median *AD* of a triangle *ABC* makes angle $\pi/6$ with side *BC* then the value of $(\cot B - \cot C)^2$ is equal to

(a) 6 (b) 9 (c) 12 (d) 15

6. If in a right angled triangle *ABC*, $4\sin A\cos B - 1 = 0$ and $\tan A$ is finite, then

(a) angles are in A.P. (b) angles are in G.P.

(c) angles are in H.P. (d) none of these

7. Let $A=\begin{bmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{bmatrix}$ and $B = A^2$.

If $(a - b)^2 + (p - q)^2 = 25$, $(b - c)^2 + (q - r)^2 = 36$, $(c - a)^2 + (r - p)^2 = 49$, then det *B* is

(a) 192 (b) 864

(c) 3456 (d) $25 \times 36 \times 47$

8. If in a ΔABC, the incircle passing through the point of intersection of perpendicular bisectors of sides *BC*, *AB* then $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=$

(a) $\sqrt{2}$ (b) $\sqrt{2}-1$ (c) $\sqrt{2}+1$ (d) $\frac{1}{2}$

9. If the lengths of medians of triangle are 2 units, 3 units and 4 units, then the area of the triangle is

(a) greater than $4\sqrt{3}$ (b) less than $4\sqrt{3}$

(c) less than or equal to $4\sqrt{3}$

(d) greater than or equal to $4\sqrt{3}$

10. In a ΔABC, $AB = AC$, *P* and *Q* are points on *AC* and *AB* respectively such that $CB = BP = PQ = QA$. If $\angle AQP = \theta$, then $\tan^2\theta$ is a root of the equation

(a) $y^3 + 21y^2 - 35y - 12 = 0$

(b) $y^3 - 21y^2 + 35y - 12 = 0$

(c) $y^3 - 21y^2 + 35y - 7 = 0$

(d) $12y^3 - 35y^2 + 35y - 12 = 0$

11. The circle $x^2 + y^2 = 16$ touches the sides *BC*, *CA* and *AB* of ΔABC respectively at *D*, *E* and *F*. If the lengths *BD*, *CE* and *AF* are consecutive integers then the largest side of the triangle is equal to

(a) 13 (b) 14

(c) 15

(d) cannot be determined

12. In a triangle *ABC*, if $\cot A : \cot B : \cot C = 30 : 19 : 6$ then the sides *a*, *b*, *c* are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

13. If two sides of a triangle are roots of the equation $x^2 - 7x + 8 = 0$ and the angle between these sides is 60° then the product of inradius and circumradius of the triangle is

(a) $\frac{8}{7}$ (b) $\frac{5}{3}$ (c) $\frac{5\sqrt{2}}{3}$ (d) 8

14. If $f(x) = \sin^{-1}(\text{cosec}(\sin^{-1}x)) + \cos^{-1}(\sec(\cos^{-1}x))$ has

(a) exactly two values (b) one value

(c) undefined (d) infinite values

15. If $\tan^{-1}\alpha = \tan^{-1}(\alpha^2 + k) + \frac{\pi}{4}$ is true for atleast one $\alpha \in [0, 2]$ and $\alpha \in I$, then number of all possible integral values of k is

(a) 0 (b) 1 (c) 2 (d) 3

16. If α is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$ ($b < c$), then the value of $\tan^{-1}\alpha + \tan^{-1}(\alpha^{-1})$ is equal to

(a) $-\pi/2$ (b) $\pi/2$

(c) 0 (d) can't be determined

17. Considering the principal values of inverse trigonometric functions $2\tan^{-1}(-3)$ is equal to

(a) $-\pi - \cos^{-1}\left(\frac{4}{5}\right)$ (b) $-\pi + \cos^{-1}\left(\frac{4}{5}\right)$

(c) $-\frac{\pi}{2} - \tan^{-1}\left(\frac{4}{3}\right)$ (d) $\frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$

18. Let x_1, x_2, x_3, x_4 be four non-zero numbers satisfying the equation $\tan^{-1}\frac{a}{x} + \tan^{-1}\frac{b}{x} + \tan^{-1}\frac{c}{x} + \tan^{-1}\frac{d}{x} = \frac{\pi}{2}$.

Then

(a) $x_1 + x_2 + x_3 + x_4 = a + b + c + d$

(b) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \neq 0$

(c) $x_1x_2x_3x_4 = abcd$

(d) $(x_2 + x_3 + x_4)(x_3 + x_4 + x_1)(x_4 + x_1 + x_2)(x_1 + x_2 + x_3) = abcd$

19. If $x \in \left(\frac{3\pi}{2}, 2\pi\right)$ then

$\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}] =$

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) 0 (d) π

20. Number of solutions to equation

$\sqrt{|\sin^{-1}|\cos x| + |\cos^{-1}|\sin x||} = \sin^{-1}|\cos x| - \cos^{-1}|\sin x|$, if $x \in [0, 4\pi]$ is

(a) 1 (b) 2 (c) 4 (d) 8

21. If $\frac{1}{2}\sin^{-1}\left(\frac{3\sin 2\theta}{5 + 4\cos 2\theta}\right) = \tan^{-1}x$, then $x =$

(a) $\frac{1}{3}\sin\theta$ (b) $\frac{1}{3}\tan\theta$

(c) $\frac{1}{3}\sin 2\theta$ (d) $\frac{1}{2}\sin^{-1}\left(\frac{\theta}{3}\right)$

22. The minimum value of $(\sec^{-1}x)^2 + (\operatorname{cosec}^{-1}x)^2$ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{8}$ (c) $\frac{\pi^2}{16}$ (d) $\frac{2\pi}{3}$

23. The value of

$\cot^{-1}\left(2^2 + \frac{1}{2}\right) + \cot^{-1}\left(2^3 + \frac{1}{2^2}\right) + \cot^{-1}\left(2^4 + \frac{1}{2^3}\right) + \ldots \infty$

(a) $\tan^{-1}2$ (b) $\cot^{-1}2$

(c) $\pi/4$ (d) none of these

24. If λ is perimeter of ΔABC then $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}$ is equal to

(a) λ (b) 2λ (c) $\lambda/2$ (d) $\lambda^2/4$

25. The sum of the infinite series

$\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots..$ is equal to

(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d) $\pi/8$

26. If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$ where $[\cdot]$ denotes the greatest integer function, then x belongs to the interval

(a) $[\tan\sin\cos 1, \tan\sin\cos\sin 1]$

(b) $[\sin\cos\tan 1, \sin\cos\sin\tan 1]$

(c) $[1, -1]$

(d) None of these

SECTION-II

Multiple Correct Answer Type

27. If area of ΔABC, Δ and angle C, are given and if the side c opposite to given angle is minimum, then

(a) $a = \sqrt{\frac{2\Delta}{\sin C}}$ (b) $b = \sqrt{\frac{2\Delta}{\sin C}}$

(c) $a = \frac{4\Delta}{\sin C}$ (d) $b = \frac{4\Delta}{\sin^2 C}$

28. If Δ represents area of acute ΔABC, then

$\sqrt{a^2b^2 - 4\Delta^2} + \sqrt{b^2c^2 - 4\Delta^2} + \sqrt{c^2a^2 - 4\Delta^2} =$

(a) $a^2 + b^2 + c^2$ (b) $\frac{a^2 + b^2 + c^2}{2}$

(c) $ab\cos C + bc\cos A + ca\cos B$

(d) $ab\sin C + bc\sin A + ca\sin B$

29. In ΔABC, the value of $c\cos(A - \theta) + a\cos(C + \theta)$ cannot exceed ($\theta \in (0, 2\pi)$) [Letters have usual meanings]

(a) a (b) b (c) c (d) s

30. If maximum and minimum values of the determinant $\Delta = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$ are α and β then

(a) $\alpha + \beta^{99} = 4$ (b) $\alpha^3 - \beta^{17} = 26$

(c) $(\alpha^{2n} - \beta^{2n})$ is always an even integer

(d) a triangle can be constructed having its sides as $\alpha - \beta$, $\alpha + \beta$, $\alpha + 3\beta$

31. In a ΔABC, which of the following statements are true?

(a) Maximum value of $\sin 2A + \sin 2B + \sin 2C$ is same as maximum value of $\sin A + \sin B + \sin C$.

(b) $R \geq 2r$, where R is circumradius and r is inradius

(c) $R^2 \geq \dfrac{abc}{a+b+c}$

(d) ΔABC is right angled if $r + 2R = s$, where s is semi perimeter.

32. Let l is the length of medians from the vertex A to the side BC of a ΔABC, then

(a) $4l^2 = 2b^2 + 2c^2 - a^2$

(b) $4l^2 = b^2 + c^2 + 2bc\cos A$

(c) $4l^2 = a^2 + 4bc\cos A$

(d) $4l^2 = (2s-a)^2 - 4bc\sin^2\dfrac{A}{2}$

33. If a right angled ΔABC of maximum area is inscribed within a circle of radius R, then Δ represents area of triangle ABC and r, r_1, r_2, r_3 represent inradius and exradii, and s is the semi perimeter of ΔABC then

(a) $\Delta = R^2$

(b) $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{\sqrt{2}+1}{R}$

(c) $r=(\sqrt{2}-1)R$ (d) $s=(1+\sqrt{2})R$

34. For $0<\theta<\dfrac{\pi}{2}$, the solution (s) of

$$\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2} \text{ is (are)}$$

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{\pi}{12}$ (d) $\dfrac{5\pi}{12}$

35. Which one of the following quantities is/are positive?

(a) $\cos(\tan^{-1}(\tan 4))$ (b) $\sin(\cot^{-1}(\cot 4))$

(c) $\tan(\cos^{-1}(\cos 5))$ (d) $\cot(\sin^{-1}(\sin 4))$

36. The values of 'x' satisfying the equation $\sin^{-1}|\sin x| = \sqrt{\sin^{-1}|\sin x|}$ are

(a) $2n\pi - 1$, $n \in Z$ (b) $2n\pi + 1$, $n \in Z$

(c) $(2n+1)\pi - 1$, $n \in Z$ (d) $(2n+1)\pi + 1$, $n \in Z$

37. If $\alpha \leq \sin^{-1}x + \cos^{-1}x - \tan^{-1}x \leq \beta$, then

(a) $\alpha = -\pi/4$ (b) $\beta = 3\pi/4$

(c) $[\alpha] = 0$ where $[.]$ denotes greatest integer function

(d) $[\alpha - \beta] = -2$ where $[.]$ denotes greatest integer function

38. For $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

(a) minimum value $= \dfrac{\pi^3}{32}$

(b) minimum value $= \dfrac{7\pi^3}{8}$

(c) maximum value $= \dfrac{7\pi^3}{8}$

(d) maximum value $= \dfrac{\pi}{2}$

SECTION-III

Comprehension Type

Paragraph for Question Nos. 39 to 41

In an acute angled triangle ABC, $a = 13 + 4\sqrt{3}$, $b = 11$ and $\sin(A-B) = 4/5$

39. Angle C is

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{3}$

(c) $\tan^{-1}2$ (d) $\tan^{-1}\left(\dfrac{2\sqrt{3}+1}{2\sqrt{3}-1}\right)$

40. Angle B is

(a) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)$ (b) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)-\dfrac{\pi}{3}$

(c) $\dfrac{\pi}{3}-\dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right)$ (d) none of these

41. Side c of the triangle ABC is

(a) $\sqrt{3}(6+\sqrt{5})$ (b) $\sqrt{6}(5+\sqrt{3})$

(c) $\sqrt{5}(\sqrt{6}+\sqrt{3})$ (d) none of these

Paragraph for Question Nos. 42 to 44

Let ABC be any triangle and P be a point inside it such that $\angle PAB = \dfrac{\pi}{18}$, $\angle PBA = \dfrac{\pi}{9}$, $\angle PCA = \dfrac{\pi}{6}$, $\angle PAC = \dfrac{2\pi}{9}$. Let $\angle PCB = x$, then

42. $x =$

(a) $\dfrac{\pi}{9}$ (b) $\dfrac{2\pi}{9}$

(c) $\dfrac{\pi}{3}$ (d) none of these

43. ΔABC is

(a) equilateral (b) isosceles

(c) scalene (d) right angled

44. Which of the following is true?

(a) $BC > AC$ (b) $AC < AB$

(c) $AC > AB$ (d) $BC = AC$

Paragraph for Question Nos. 45 to 47

Let ΔABC be any triangle and let D, E, F feet of perpendicular from vertices A, B, C on opposite side BC, CA, AB respectively. Then the ΔDEF is known as pedal Δ of ABC. H is orthocentre of the ΔABC. We note that $\angle HDC = \angle HEC = 90°$, so the points H, D, C and E are concyclic.

In question, triangle ABC is to be assumed an acute angled triangle. In case ΔABC be obtuse angled with A as obtuse angle. The angle of pedal Δ will be $2A - 180°$, $2B$, $2C$ and side will be represented by $a\cos A$, $b\cos B$, $c\cos C$.

45. If l, m, n denotes the side of a pedal triangle, then $\frac{l}{a^2}+\frac{m}{b^2}+\frac{n}{c^2} =$

(a) $\frac{a^2+b^2+c^2}{a^3+b^3+c^3}$ (b) $\frac{a^2+b^2+c^2}{2abc}$

(c) $\frac{a^3+b^3+c^3}{abc(a+b+c)}$ (d) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

46. If R be circumradius of a Δ then circumradius of a pedal Δ is

(a) R (b) $\frac{2R}{3}$ (c) $\frac{R}{3}$ (d) $\frac{R}{2}$

47. The inradius of pedal Δ of a ΔABC

(a) $\frac{R}{2}$ (b) $R\sin A\sin B\sin C$

(c) $2R\cos A\cos B\cos C$ (d) $4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Paragraph for Question Nos. 48 to 50

Let $a = \cos^{-1}(\cos 20)$, $b = \cos^{-1}(\cos 30)$ and $c = \sin^{-1}(\sin(a+b))$ then

48. Maximum value of $\sin(2(a+b+c)x) + \cos^2((a+b+c)x)$ is

(a) $\frac{\sqrt{5}+1}{2}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\frac{\sqrt{3}-1}{2}$ (d) none of these

49. The largest integer x for which $\sin^{-1}(\sin x) \geq |x - (a+b+c)|$ is

(a) 1 (b) 2 (c) 3 (d) 4

50. If $5\sec^{-1}x + 10\sin^{-1}y = 10(a+b+c)$ then the value of $\tan^{-1}x + \cos^{-1}(y-1)$ is

(a) $\pi/2$ (b) $\pi/4$ (c) π (d) 0

Paragraph for Question Nos. 51 to 53

The trigonometric equation is $\sin x + 3\sin 2x + \sin 3x = \cos x + 3\cos 2x + \cos 3x$ when x lies in first four quadrants. It means $x \in [0, 2\pi]$, then

51. How many solutions are there?

(a) 2 (b) 3 (c) 4 (d) 5

52. The difference between greatest and least solution of x is

(a) $\frac{3\pi}{2}$ (b) $\frac{\pi}{2}$ (c) π (d) 10π

53. The sum of the solutions of x is

(a) $\frac{14\pi}{23}$ (b) $\frac{7\pi}{2}$ (c) $\frac{9\pi}{8}$ (d) 6π

SECTION-IV

Matrix-Match Type

54. In triangle ABC if $2a^2 + 4b^2 + c^2 = 4ab + 2ac$ where a, b, c are the sides of a triangle then, match the following:

Column-I		Column-II	
(A)	$\cos A$	(p)	1/4
(B)	$\cos B$	(q)	7/8
(C)	$\cos C$	(r)	0
(D)	$\sin(A - C)$	(s)	1

55. r, r_1, r_2, r_3 are the inradii and exradii of ΔABC and r_1, r_2, r_3 are the roots of $x^3 - 11x^2 + 36x - 36 = 0$ then match the following:

Column-I		Column-II	
(A)	$\cos A + \cos B + \cos C =$	(p)	6
(B)	$\sin A + \sin B + \sin C =$	(q)	30
(C)	If I_1, I_2, I_3 are the excentres of ΔABC then area of $\Delta I_1I_2I_3$ (in sq. units) is	(r)	7/5
(D)	Area of ΔABC (in sq. units) is	(s)	12/5

56. Let (x, y) be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$$

Column-I		Column-II	
(A)	If $a = 1$, and $b = 0$, then (x, y)	(p)	lies on the circle $x^2 + y^2 = 1$
(B)	If $a = 1$ and $b = 1$, then (x, y)	(q)	lies on $(x^2 - 1)(y^2 - 1) = 0$
(C)	If $a = 1$ and $b = 2$, then (x, y)	(r)	lies on $y = x$
(D)	If $a = 2$ and $b = 2$, then (x, y)	(s)	lies on $(4x^2 - 1)(y^2 - 1) = 0$

57. Match the following Column-I with Column-II in a strict sense of interval.

	Column-I		Column-II
(A)	If $x^2 - 4x - 2\pi \le \sin^{-1}(\sin 5)$ then 'x' lies in	(p)	$R - [-1, 1]$
(B)	If $f(x) = x^2 - 4$ and $g(x) = f(x) + 5$ such that $f(\lvert g(x)\rvert) > 0$ then 'x' lies in	(q)	$R - (-1, 1)$
(C)	If $3\sin\theta + 4\cos\theta = \dfrac{5(2a-3)}{2-3a}$ has a real solution for θ then 'a' lies in	(r)	$[-4, 4]$
(D)	If $4x^4 + 9y^4 = 64$ and $z = \sqrt{3}xy$ then 'z' lies in	(s)	$[-1, 5]$

SECTION-V

Integer Answer Type

58. If a, b, c, d are in A.P. and $\int_0^4 f(t)dt = -32$ where

$$f(t) = \begin{vmatrix} t+a & t+b & t+a-c \\ t+b & t+c & t-1 \\ t+c & t+d & t-b+d \end{vmatrix}$$

then the positive common difference of A.P. is

59. In a triangle ABC, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is p/q then $p - q =$

60. If in a triangle ABC, the line containing circumcentre and orthocentre is parallel to AC, then $\tan A \tan C =$

61. The number of solutions, of the pair of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is

62. In ΔABC, if BC is unity,

$$\sin\frac{A}{2} = x_1, \sin\frac{B}{2} = x_2, \cos\frac{A}{2} = x_3 \text{ and } \cos\frac{B}{2} = x_4 \text{ with } \left(\frac{x_1}{x_2}\right)^{2007} - \left(\frac{x_3}{x_4}\right)^{2006} = 0,$$

then the length of AC is

63. If $\cos^{-1}(4x^3 - 3x) = a + b\cos^{-1}x$ for $-1 < x < -\frac{1}{2}$. Then $\lvert[a + b + 2]\rvert =$ ___ where [.] is G.I.F.

64. The least value of n for which $(n-2)x^2 + 8x + n + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12) \forall x \in R$ where $n \in N$, is

65. The number of positive integral solutions of the equation $\tan^{-1}x + \cos^{-1}\dfrac{y}{\sqrt{1+y^2}} = \sin^{-1}\dfrac{3}{\sqrt{10}}$ are

66. Find the integral value of the area bounded by the function $f(x) = (\cos^{-1}(\cos x))^2$ and $g(x) = \cos^{-1}(\cos\lvert x\rvert)$ and $\lvert x\rvert = 2\pi$.

67. For $x, y, z, t \in R$ if $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \ge t^2 - \sqrt{2\pi t} + 3\pi$, then the value of $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z + \tan^{-1}\left(\sqrt{\dfrac{2}{\pi}}t\right)$ is

SOLUTIONS

1. (a): $x + k + y = 16$.

Also $(t+7) \times t = 2 \times (2+13)$

$\Rightarrow t = 3$

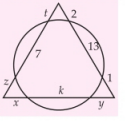
Now, $x(x + k) = 6 \times (6 + 7)$

and $y(y + k) = 1 \times 14$

Solving we get

$x = 10 - \sqrt{22}, y = 6 - \sqrt{22}$,

$k = 2\sqrt{22}$



2. (a): Let $\alpha = \dfrac{\pi - A}{4}, \beta = \dfrac{\pi - B}{4}, \gamma = \dfrac{\pi - C}{4}$

$\Rightarrow \alpha + \beta + \gamma = \dfrac{\pi}{2} \Rightarrow \sum \tan\alpha \tan\beta = 1$

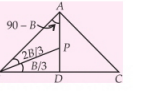
$\sum \tan^2\alpha = 1 = \sum \tan\alpha \tan\beta$ $(a^2 + b^2 + c^2 = ab + bc + ca$

$\Rightarrow a = b = c$, where a, b, c are elements of $R)$

$\Rightarrow \tan\alpha = \tan\beta = \tan\gamma$

3. (c): $\dfrac{AP}{\sin\dfrac{2B}{3}} = \dfrac{c}{\sin\left(90 + \dfrac{B}{3}\right)}$

$\Rightarrow AP = 2c\sin\dfrac{B}{3}$



4. (b): $\dfrac{\cot A + \cot C}{\cot B} = \dfrac{\sin(A+C)\sin B}{\sin A \sin C \sin B}$

$= \dfrac{\sin^2 B}{\sin A \cos B \sin C}$ $[\because A + B + C = \pi]$

$= \dfrac{4R^2b^2}{4R^2ac\cos B} = \dfrac{2b^2}{2ac\cos B} = \dfrac{2b^2}{a^2 + c^2 - b^2}$

$= \dfrac{2b^2}{2002b^2 - b^2} = \dfrac{2}{2001}$

5. (c): Applying mn theorem

$(BD + DC)\cot\dfrac{\pi}{6} = DC\cot B - BD\cot C$

$\Rightarrow$ $(\cot B - \cot C)^2 = 12$.

6. (a) : $4\sin A\cos B = 1$, so A and B cannot be $\pi/2$

[as if $B = \frac{\pi}{2}$, then $\cos B = 0$ and if $A = \frac{\pi}{2}$, $\tan A$ is not defined]

$\Rightarrow \quad C = \frac{\pi}{2}, B = \frac{\pi}{2} - A \;\Rightarrow\; 4\sin A\cos\left(\frac{\pi}{2} - A\right) = 1$

$\Rightarrow \quad \sin^2 A = \frac{1}{4}$

$\Rightarrow \quad \sin A = \frac{1}{2} \;\Rightarrow\; A = \frac{\pi}{6} \;\Rightarrow\; B = \frac{\pi}{3}$

so angles are in A.P.

7. (b): $\det A$ is twice the area of the triangle with vertices (a, p), (b, q), (c, r) with sides 5, 6, 7

$\Delta^2 = s(s-a)(s-b)(s-c) \;\Rightarrow\; 16\Delta^2 = 18 \cdot 8 \cdot 6 \cdot 4$

$\det B = (\det A)^2 = 4\Delta^2 = 18 \cdot 8 \cdot 6 = 864$

8. (b): Given that the incircle passes through the circumcentre of ΔABC. Therefore the distance between circumcentre, and incentre $= \sqrt{R^2 - 2rR} = r$

$\Rightarrow \quad r^2 + 2rR - R^2 = 0$

$\Rightarrow \quad \left(\frac{r}{R}\right)^2 + 2\frac{r}{R} - 1 = 0 \;\Rightarrow\; \frac{r}{R} = \sqrt{2} - 1$ and

$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

9. (b): If length of medians be m_a, m_b, m_c then

$2(m_a + m_b + m_c) < 3(a+b+c) < 4(m_a + m_b + m_c)$

$\Rightarrow \quad 6 < a + b + c < 12 \;\Rightarrow\; s < 6$

Also,

$[(s-a)(s-b)(s-c)]^{1/3} < \frac{s-a+s-b+s-c}{3} < 2$

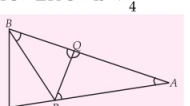
$\therefore \quad (s-a)(s-b)(s-c) < 8$

$\therefore \quad s(s-a)(s-b)(s-c) < 48 \quad \Rightarrow \quad \Delta < 4\sqrt{3}$

10. (c) : $\angle QAP = \angle QPA = 90° - \frac{\theta}{2}$

$\angle PQB = \angle PBQ = 180° - \theta$

$\angle BCA = \angle ABC = \angle BPC = 45° + \frac{\theta}{4}$



Now $\left(90° - \frac{\theta}{2}\right) + (2\theta - 180°) + \left(45° + \frac{\theta}{4}\right) = 180° \;\Rightarrow\; \theta = \frac{5\pi}{7}$

$\Rightarrow \quad 7\theta = 5\pi \;\Rightarrow\; 4\theta = 5\pi - 3\theta \;\Rightarrow\; \tan 4\theta = -\tan 3\theta$

$\Rightarrow \quad \frac{2\tan 2\theta}{1 - \tan^2 2\theta} = -\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$

$\Rightarrow \quad \frac{2\left[\frac{2t}{1-t^2}\right]}{1 - \left(\frac{2t}{1-t^2}\right)^2} = -\left(\frac{3t - t^3}{1 - 3t^2}\right)$, where $t = \tan\theta$

$\Rightarrow \quad \frac{4(1-t^2)}{(1-t^2)^2 - 4t^2} = \frac{t^2 - 3}{1 - 3t^2}$

$\Rightarrow \quad 4(1 - 4t^2 + 3t^4) = (t^2 - 3)(1 - 6t^2 + t^4)$

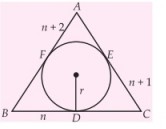
$\Rightarrow \quad t^6 - 21t^4 + 35t^2 - 7 = 0$

$\therefore \quad \tan^2\theta$ is the root of the equation

$y^3 - 21y^2 + 35y - 7 = 0$

11. (c) : Let $BD = n$, $CE = n + 1$, $AF = n + 2$.

Then $BD = BF = n$, $CE = CD = n + 1$, $AF = AE = n + 2$



$\therefore \quad a = BC = 2n + 1$, $b = 2n + 3$, $c = 2n + 2$, $s = 3n + 3$

$r = \frac{\Delta}{s} = \frac{\sqrt{(3n+3)(n+2)n(n+1)}}{3n+3}$

$\therefore \quad 4 = \sqrt{\frac{(n+2)n}{3}} \;\Rightarrow\; n(n+2) = 48 \Rightarrow n = 6$

$\therefore$ the largest side of the triangle is $2n + 3 = 15$.

12. (a) : $\cot A = \frac{\cos A}{\sin A} = \frac{\frac{b^2 + c^2 - a^2}{2bc}}{\frac{a}{2R}} = \left(\frac{b^2 + c^2 - a^2}{abc}\right)R$

$\therefore \quad \cot A : \cot B : \cot C = b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2$

Clearly if $\cot A : \cot B : \cot C = l : m : n$

then $a : b : c = \sqrt{m+n} : \sqrt{n+l} : \sqrt{l+m}$

$= \sqrt{19+6} : \sqrt{6+30} : \sqrt{30+19} = 5:6:7$

$\therefore \quad a, b, c$ are in A.P.

13. (b): Let a and b be the roots of $x^2 - 7x + 8 = 0$. Then $a + b = 7$, $ab = 8$

Also $C = 60°$

Now, $\cos 60° = \frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$

$\Rightarrow \quad ab = (a + b)^2 - 2ab - c^2$

$\therefore \quad c^2 = (a+b)^2 - 3ab = 49 - 24 = 25 \Rightarrow c = 5$

Thus $r \cdot R = \dfrac{abc}{2(a+b+c)} = \dfrac{8\times 5}{2(7+5)} = \dfrac{5}{3}$

14. (b): Clearly $f(x)$ is defined only for $x = \pm 1$

and $f(1) = f(-1) = \dfrac{\pi}{2}$

15. (b): We have $k = 1 - \dfrac{2}{\alpha+1} - \alpha^2$ gives $k = -1$ only.

16. (a): Let $f(x) = x^3 + bx^2 + cx + 1$
so, $f(0) = 1 > 0$, $f(-1) = b - c < 0$

so, $-1 < \alpha < 0$, therefore $\tan^{-1}(\alpha) + \tan^{-1}\left(\dfrac{1}{\alpha}\right) = -\dfrac{\pi}{2}$

17. (c): Let $\theta = 2\tan^{-1}(-3) \Rightarrow -\pi < \theta < 0$

$$\tan\frac{\theta}{2} = -3 \Rightarrow \tan\theta = \frac{2\tan\left(\dfrac{\theta}{2}\right)}{1-\tan^2\left(\dfrac{\theta}{2}\right)} = \frac{3}{4}$$

$\Rightarrow \quad -\pi < \theta < -\dfrac{\pi}{2}$

Let $\alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$ (principal value) then $\theta = \alpha - \pi$

Now $\cos\alpha = \dfrac{4}{5} \Rightarrow \theta = \cos^{-1}\left(\dfrac{4}{5}\right) - \pi = \dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{4}{5}\right) - \pi$

$$= -\sin^{-1}\left(\frac{4}{5}\right) - \frac{\pi}{2}$$

$\cot\alpha = \dfrac{4}{3} \Rightarrow \theta = \cot^{-1}\left(\dfrac{4}{3}\right) - \pi = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{4}{3}\right) - \pi$

$$= -\tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{2}$$

18. (d): $\tan^{-1}\dfrac{a}{x} + \tan^{-1}\dfrac{b}{x} = \dfrac{\pi}{2} - \left(\tan^{-1}\dfrac{c}{x} + \tan^{-1}\dfrac{d}{x}\right)$

$$\Rightarrow \quad \tan^{-1}\left(\frac{\dfrac{a}{x}+\dfrac{b}{x}}{1-\dfrac{ab}{x^2}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\dfrac{c}{x}+\dfrac{d}{x}}{1-\dfrac{cd}{x^2}}\right)$$

$$\Rightarrow \quad \tan^{-1}\frac{(a+b)x}{x^2-ab} = \cot^{-1}\frac{(c+d)x}{x^2-cd}$$

$$\Rightarrow \quad \frac{(a+b)x}{x^2-ab} = \frac{x^2-cd}{(c+d)x}$$

$$\Rightarrow \quad x^4 - (ab+ac+ad+bc+bd+cd)x^2 + abcd = 0$$

Its roots are x_1, x_2, x_3, x_4

$\therefore \quad \sum x_1 = 0, \ \sum x_1x_2 = -\sum ab, \ \sum x_1x_2x_3 = 0$

and $x_1x_2x_3x_4 = abcd$

$\therefore \quad \sum \dfrac{1}{x_1} = 0$ and

$$(x_2+x_3+x_4)(x_3+x_4+x_1)(x_4+x_1+x_2)(x_1+x_2+x_3) = x_1x_2x_3x_4$$

19. (b): For given range of 'x',
$\cos^{-1}(\cos x) + \sin^{-1}(\sin x) = 2\pi - x + x - 2\pi = 0$

20. (c): Let's find a solution for $x \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

Case-I : $x \in \left[0, \dfrac{\pi}{2}\right] \Rightarrow \dfrac{\pi}{2} - x \in \left[0, \dfrac{\pi}{2}\right]$

$$\therefore \quad \sqrt{\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right) + \cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)} = 0$$

$$\Rightarrow \quad \sqrt{2\left(\frac{\pi}{2}-x\right)} = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$$

Case-II : $x \in \left[-\dfrac{\pi}{2}, 0\right) \Rightarrow \dfrac{\pi}{2} - x \in \left(\dfrac{\pi}{2}, \pi\right]$

$$= \sqrt{\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right) + \cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)}$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right) - \cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)$$

$\Rightarrow \quad \sqrt{\pi + 2x} = 0 \Rightarrow x = -\dfrac{\pi}{2}$; using periodicity we conclude that there are four solutions.

21. (b): $\dfrac{3\sin 2\theta}{5+4\cos 2\theta} = \dfrac{6\tan\theta}{9+\tan^2\theta} = \dfrac{2\left(\frac{1}{3}\tan\theta\right)}{1+\left(\frac{1}{3}\tan\theta\right)^2}$

$$= \frac{2\tan\alpha}{1+\tan^2\alpha}$$

where $\tan\alpha = \dfrac{1}{3}\tan\theta \Rightarrow \alpha = \tan^{-1}\left(\dfrac{1}{3}\tan\theta\right)$

$$\frac{1}{2}\sin^{-1}(\sin 2\alpha) = \alpha \Rightarrow x = \frac{1}{3}\tan\theta$$

22. (b): $I = (\sec^{-1}x)^2 + (\operatorname{cosec}^{-1}x)^2$

$= (\sec^{-1}x + \operatorname{cosec}^{-1}x)^2 - 2\sec^{-1}x\,\operatorname{cosec}^{-1}x$

$= \dfrac{\pi^2}{4} - 2\sec^{-1}x\left(\dfrac{\pi}{2} - \sec^{-1}x\right) = \dfrac{\pi^2}{4} + 2(\sec^{-1}x)^2 - \pi\sec^{-1}x$

$$= \frac{\pi^2}{4} + 2\left[(\sec^{-1}x)^2 - 2\times\frac{\pi}{4}\sec^{-1}x + \left(\frac{\pi}{4}\right)^2\right] - \frac{\pi^2}{8}$$

$$= 2\left(\sec^{-1}x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} \geq \frac{\pi^2}{8}$$

23. (b): $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\tan^{-1}\left(\frac{2^n}{1+2^{2n+1}}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}(\tan^{-1}(2^{n+1}) - \tan^{-1}(2^n))$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}(\tan^{-1}(2^{n+1} - \tan^{-1}2))$$

24. (c): $b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2} = b\,\frac{s(s-c)}{ab} + c\,\frac{s(s-b)}{ac}$

$$= \frac{s}{a}(s-c+s-b) = s = \frac{\lambda}{2}$$

25. (b): The nth term of the series can be written as

$u_n = \cot^{-1}(2n^2) = \tan^{-1}(1/2n^2)$

$$= \tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

So that $u_1 = \tan^{-1}3 - \tan^{-1}1$

$u_2 = \tan^{-1}5 - \tan^{-1}3$

$u_3 = \tan^{-1}7 - \tan^{-1}5$

- - - - - -

and the sum to n terms is therefore written as

$S_n = \tan^{-1}(2n+1) - \tan^{-1}1$

$$\Rightarrow \lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

26. (a): We have, $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$

$$\Rightarrow 1 \leq \sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x \leq \frac{\pi}{2}$$

$$\Rightarrow \sin 1 \leq \cos^{-1}\sin^{-1}\tan^{-1}x \leq 1$$

$$\Rightarrow \cos\sin 1 \geq \sin^{-1}\tan^{-1}x \geq \cos 1$$

$$\Rightarrow \sin\cos\sin 1 \geq \tan^{-1}x \geq \sin\cos 1$$

$$\Rightarrow \tan\sin\cos\sin 1 \geq x \geq \tan\sin\cos 1$$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$

27. (a, b): $c^2 = a^2 + b^2 - 2ab\cos C$

$= (a-b)^2 + 2ab(1-\cos C)$ $= (a-b)^2 + \frac{4\Delta}{\sin C}(1-\cos C)$

Hence for c to be minimum $a = b$

Also, $\Delta = \frac{1}{2}ab\sin C \Rightarrow a^2 = \frac{2\Delta}{\sin C} = b^2$

28. (b, c): $2\Delta = ab\sin C \Rightarrow a^2b^2 - 4\Delta^2 = a^2b^2\cos^2 C$

29. (b, d): $c\cos(A-\theta) + a\cos(C+\theta)$

$= \cos\theta(c\cos A + a\cos C) + \sin\theta(c\sin A - a\sin C)$

$= b\cos\theta \leq b$

30. (a, b, c): $C_1 \to C_1 + C_2 \Rightarrow \begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 2 & 1+\cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1+\sin 2x \end{vmatrix}$

$= 2 + \sin 2x$; $\therefore\ \alpha = 3,\ \beta = 1$

31. (a, b, c, d): (a) Maximum value of

$\sin 2A + \sin 2B + \sin 2C = \frac{3\sqrt{3}}{2}$ when $A = B = C = \frac{\pi}{3}$

$\sin A + \sin B + \sin C$ maximum value is same.

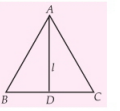
(b) $\frac{r}{R} = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq 4.\frac{1}{8} \Rightarrow R \geq 2r$

(c) $\frac{abc}{a+b+c} = \frac{4R\Delta}{2s} = 2R\cdot r \leq R^2$

(d) $r = s - 2R \Rightarrow \angle A$ or $\angle B$ or $\angle C$ is $\pi/2$

(for if $\angle A = 90°$, $2R = a$ then $r = (s-a)\tan\frac{A}{2} = s-a$)

32. (a, b, c, d): D is the mid point



$$\Rightarrow AB^2 + AC^2 = 2(AD^2 + BD^2)$$

$$c^2 + b^2 = 2\left[l^2 + \left(\frac{a}{2}\right)^2\right]$$

$$\Rightarrow c^2 + b^2 = 2l^2 + \frac{a^2}{2}$$

$$\Rightarrow 4l^2 = 2b^2 + 2c^2 - a^2$$

$= b^2 + c^2 + (b^2 + c^2 - a^2) = b^2 + c^2 + 2bc\cos A$

$= (b^2 + c^2 - a^2) + a^2 + 2bc\cos A$

$= 2bc\cos A + a^2 + 2bc\cos A = 4bc\cos A + a^2$

33. (a, b, c, d): For a right angled triangle inscribed in a circle of radius R the length of the hypotenuse is $2R$

Then area is maximum when it is an isosceles triangle with each equal side $= \sqrt{2}R$

$$\therefore\ s = \frac{1}{2}(2\sqrt{2}+2)R = (\sqrt{2}+1)R$$

$$\Delta = \frac{1}{2}\sqrt{2}R\cdot\sqrt{2}R = R^2$$

$$r = \frac{\Delta}{s} = \frac{R^2}{(\sqrt{2}+1)R} \Rightarrow r = (\sqrt{2}-1)R$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r} = \frac{1}{(\sqrt{2}-1)R} = \frac{\sqrt{2}+1}{R}$$

34. (c, d): We have

$$\sum_{m=1}^{6}\operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\frac{\pi}{4}}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\left[\left(\theta+\frac{m\pi}{4}\right)-\left(\theta+\frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \sum_{m=1}^{6} \left[\frac{\sin\left(\theta+\frac{m\pi}{4}\right)\cos\left(\theta+\frac{(m-1)\pi}{4}\right)}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)} - \frac{\cos\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{(m-1)\pi}{4}\right)}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}\right] = 4$$

$$\Rightarrow \sum_{m=1}^{6}\left[\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \left[\cot\theta-\cot\left(\theta+\frac{\pi}{4}\right)\right]+\left[\cot\left(\theta+\frac{\pi}{4}\right)-\cot\left(\theta+\frac{2\pi}{4}\right)\right]+....$$

$$.....+\left[\cot\left(\theta+\frac{5\pi}{4}\right)-\cot\left(\theta+\frac{6\pi}{4}\right)\right] = 4$$

$$\Rightarrow \cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right) = 4 \Rightarrow \cot\theta+\tan\theta = 4$$

$$\Rightarrow \cos^2\theta+\sin^2\theta = 4\sin\theta\cos\theta$$

$$\Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

35. (a, b, c) : (a) $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi)$
$= \cos(\pi-4) = -\cos 4 > 0$

(b) $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$
(as $\sin 4 < 0$)

(c) $\tan(\cos^{-1}(\cos(2\pi-5))) = \tan(2\pi-5) = -\tan 5 > 0$
(as $\tan 5 < 0$)

(d) $\cot(\sin^{-1}(\sin(\pi-4))) = \cot(\pi-4) = -\cot 4 < 0$

36. (a, b, c, d) : $\sin^{-1}|\sin x| = 0$ or 1

$\sin^{-1}|\sin x| = 0 \Rightarrow |\sin x| = \sin 0 \Rightarrow x = n\pi$

$\sin^{-1}|\sin x| = 1 \Rightarrow |\sin x| = \sin 1$

$\Rightarrow \sin x = \pm \sin 1$

If $\sin x = \sin 1 \Rightarrow x = 2n\pi + 1$ or $2n\pi + \pi - 1$

If $\sin x = -\sin 1 \Rightarrow x = 2n\pi - 1$ or $2n\pi + \pi + 1$

37. (b, c, d) : If $\alpha \le \frac{\pi}{2} - \tan^{-1} x \le \beta \ \forall\ x \in [-1,1]$

$-1 \le x \le 1 \Rightarrow -\frac{\pi}{4} \le -\tan^{-1} x \le \frac{\pi}{4}$

$\Rightarrow \frac{\pi}{4} \le \frac{\pi}{2} - \tan^{-1} x \le \frac{3\pi}{4}$

38. (a, c) : $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}x\left(\frac{\pi}{2}-\sin^{-1}x\right)\right]$$

$$= \frac{3\pi}{2}\left[\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{16}\right]$$

39. (b) : $\sin(A-B) = \frac{4}{5} \Rightarrow \frac{2\tan\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$

On solving, we get,

$\Rightarrow \tan\frac{(A-B)}{2} = 2$ or $\tan\frac{(A-B)}{2} = \frac{1}{2}$

Because triangle is acute angled, so $\tan\frac{(A-B)}{2} = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2+4\sqrt{3}}{24+4\sqrt{3}}$$

$$\Rightarrow \frac{1}{2} = \frac{2+4\sqrt{3}}{24+4\sqrt{3}}\cot\frac{C}{2} \Rightarrow \cot\frac{C}{2} = \sqrt{3}$$

$\Rightarrow C = 60°$

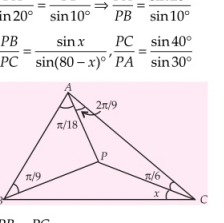
40. (c) : $C = 60° \Rightarrow A + B = 120°$ and $A - B = 2\tan^{-1}\frac{1}{2}$

$\Rightarrow B = 60° - \frac{1}{2}\tan^{-1}\frac{4}{3}$

41. (d) : Side c can be found using cosine rule viz $c^2 = a^2 + b^2 - 2ab\cos C$

42. (a) : $\frac{PA}{\sin 20°} = \frac{PB}{\sin 10°} \Rightarrow \frac{PA}{PB} = \frac{\sin 20°}{\sin 10°}$

Similarly, $\frac{PB}{PC} = \frac{\sin x}{\sin(80-x)°}$, $\frac{PC}{PA} = \frac{\sin 40°}{\sin 30°}$



$$\frac{PA}{PB}\times\frac{PB}{PC}\times\frac{PC}{PA} = 1$$

$$\Rightarrow \frac{\sin 20°}{\sin 10°}\times\frac{\sin x}{\sin(80-x)°}\times\frac{\sin 40°}{\sin 30°} = 1$$

$\Rightarrow \sin(80-x)° = 4\cos 10°\sin 40°\sin x = 2$

$\sin 50 \sin x + \sin x$

$\Rightarrow \sin(80 - x)^\circ - \sin x = 2 \sin 50^\circ \sin x$

$\Rightarrow 2 \sin(40 - x)^\circ \cos 40^\circ = 2 \sin 50^\circ \sin x$

$\Rightarrow \sin(40 - x)^\circ = \sin x \Rightarrow x = 20^\circ$

43. (b) : $\angle A = \angle C = 50^\circ$

44. (c) : $\angle ABC = 80^\circ \Rightarrow AC$ is the longest side

45. (b) : $l = a \cos A$, $m = b \cos B$, $n = c \cos C$

$$\frac{l}{a^2} + \frac{m}{b^2} + \frac{n}{c^2} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$$

46. (d) : Circumradius $= \dfrac{EF}{2\sin(\angle FDE)}$

$$= \frac{R \sin 2A}{2 \sin(180^\circ - 2A)} = \frac{R}{2}$$

47. (c) : Area of pedal $\Delta = \dfrac{1}{2}(DE)DF \sin(\angle EDF)$

$$= \frac{1}{2}(R \sin 2B)(R \sin 2C) \sin(180^o - 2A)$$

$$= \frac{1}{2} R^2 \sin 2A \sin 2B \sin 2C$$

Semi-perimeter of pedal Δ

$$s = \frac{1}{2} R(\sin 2A + \sin 2B + \sin 2C)$$

Inradius $= \dfrac{\Delta}{s} = 2R \cos A \cos B \cos C$.

48. (a) : $a = 20 - 6\pi$, $b = 10\pi - 30$,

$c = \sin^{-1} \sin(4\pi - 10) = 10 - 3\pi$

so $a + b + c = \pi$

$f(x) = \sin(2(a + b + c)x) + \cos^2((a + b + c)x)$

$$= \sin 2\pi x + \cos^2 \pi x = \sin 2\pi x + \frac{1 + \cos 2\pi x}{2}$$

$\Rightarrow$ maximum value of $f(x) = \dfrac{\sqrt{5} + 1}{2}$.

49. (c) : $\sin^{-1} \sin x \geq |x - \pi| \Rightarrow x \in \left[\dfrac{\pi}{2}, \pi\right]$

So largest integer is $x = 3$.

50. (b) : $5 \sec^{-1} x + 10 \sin^{-1} y = 10\pi$

$\Rightarrow \sec^{-1} x = \pi$ and $\sin^{-1} y = \dfrac{\pi}{2} \Rightarrow x = -1, y = 1$

So $\tan^{-1}(-1) + \cos^{-1}(0) = -\dfrac{\pi}{4} + \dfrac{\pi}{2} = \dfrac{\pi}{4}$

51. (c) : Apply $\sin C + \sin D$ & $\cos C + \cos D$ then take common & solve

$(\sin 2x - \cos 2x)(2 \cos x + 3) = 0$

$\tan 2x = 1, \cos x = \dfrac{-3}{2}$ (Rejected)

$$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

52. (a) : $\dfrac{13\pi}{8} - \dfrac{\pi}{8} = \dfrac{12\pi}{8} = \dfrac{3\pi}{2}$

53. (b) : $\dfrac{28\pi}{8} = \dfrac{7\pi}{2}$

54. A - p; B - q; C - p; D - r

The given relation can be written as

$(a - 2b)^2 + (a - c)^2 = 0$

$\Rightarrow a = 2b, a = c \Rightarrow b = \dfrac{a}{2}, c = a$

Now, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{\frac{a^2}{4} + a^2 - a^2}{2 \times \frac{a}{2} \times a} = \dfrac{1}{4}$

Similarly, $\cos B = \dfrac{7}{8}, \cos C = \dfrac{1}{4}$

$\because \quad A - C = 0 \Rightarrow \sin(A - C) = 0$

55. A - r ; B - s ; C - q ; D - p

$$\frac{1}{r} = \sum \frac{r_1 r_2}{r_1 r_2 r_3} = 1 \Rightarrow r = 1$$

$$R = \frac{1}{4}[r_1 + r_2 + r_3 - r] = \frac{1}{4}[11 - 1] = \frac{5}{2}$$

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = 1 + \frac{2}{5} = \frac{7}{5}$$

$\sum r_1 r_2 = s^2 \Rightarrow s^2 = 36 \Rightarrow s = 6$

$\Delta = rs = 6$

$\therefore \sin A + \sin B + \sin C = \dfrac{2s}{2R} = \dfrac{6}{5/2} = \dfrac{12}{5}$

$\Delta I_1 I_2 I_3 = 2Rs = 5 \times 6 = 30$

56. A - p ; B - q ; C - p ; D - s

$\sin^{-1}(ax) + \cos^{-1} y + \cos^{-1}(bxy) = \dfrac{\pi}{2}$

$\Rightarrow \cos^{-1} y + \cos^{-1}(bxy) = \dfrac{\pi}{2} - \sin^{-1}(ax) = \cos^{-1}(ax)$

Let $\cos^{-1} y = \alpha, \cos^{-1}(bxy) = \beta, \cos^{-1}(ax) = \gamma$

$\Rightarrow \quad y = \cos\alpha, bxy = \cos\beta, ax = \cos\gamma$

$\therefore$ We get $\alpha+\beta=\gamma$ and $\cos\beta = bxy$

$\Rightarrow \cos(\gamma-\alpha) = bxy \Rightarrow \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = bxy$

$\Rightarrow axy + \sin\gamma\sin\alpha = bxy \Rightarrow (a-b)xy = -\sin\alpha\sin\gamma$

$\Rightarrow (a-b)^2x^2y^2 = \sin^2\alpha\sin^2\gamma = (1-\cos^2\alpha)(1-\cos^2\gamma)$

$\Rightarrow (a-b)^2x^2y^2 = (1-a^2x^2)(1-y^2)$ ...(1)

For $a = 1$, $b = 0$, equation (1) reduces to

$x^2y^2 = (1-x^2)(1-y^2) \Rightarrow x^2+y^2 = 1$

For $a = 1$, $b = 1$, equation (1) becomes

$(1-x^2)(1-y^2) = 0 \Rightarrow (x^2-1)(y^2-1) = 0$

For $a = 1$, $b = 2$, equation (1) reduces to

$x^2y^2 = (1-x^2)(1-y^2) \Rightarrow x^2+y^2 = 1$

For $a = 2$, $b = 2$, equation (1) reduces to

$0 = (1-4x^2)(1-y^2) \Rightarrow (4x^2-1)(y^2-1) = 0$

57. A - s; B - p; C - q; D - r

(A) $x^2-4x-2\pi \le \sin^{-1}\sin(5-2\pi)$

($(5-2\pi)$ is an angle in the 4th quadrant)

or $x^2-4x-2\pi \le 5-2\pi$

$\Rightarrow x^2-4x-5 \le 0 \Rightarrow (x-5)(x+1) \le 0$

$\Rightarrow x \in [-1, 5]$

(B) $f\{g(x)\} = f(x^2-4+5) = f(x^2+1) = (x^2+1)^2-4$

$= (x^2+3)(x^2-1)$

$\because$ $f\{g(x)\} > 0 \Rightarrow x^2-1 > 0$

$\Rightarrow x < -1$ or $x > 1 \Rightarrow x \in R - [-1, 1]$

(C) $-5 \le 3\sin\theta + 4\cos\theta \le 5$

So, $-5 \le \dfrac{5(2a-3)}{2-3a} \le 5$

$\therefore -1 \le \dfrac{2a-3}{2-3a} \le 1 \Rightarrow a \in R - (-1, 1)$

(D) Let $x^2 = 4\cos\theta$ and $y^2 = \dfrac{8}{3}\sin\theta$, which satisfy $4x^4 + 9y^4 = 64$

$\therefore z^2 = 3x^2y^2 = 3(4\cos\theta)\left(\dfrac{8}{3}\sin\theta\right) = 16\sin2\theta \le 16$

$\therefore z^2 - 16 \le 0 \Rightarrow -4 \le z \le 4$

58. (2) : $d = \pm 2$, +ve, $d = 2$

59. (9) : $a = 4k$, $b = 5k$, $c = 6k$

$s = \dfrac{15}{2}k,\ s-a = \dfrac{7}{2}k,\ s-b = \dfrac{5}{2}k, s-c = \dfrac{3}{2}k$

$S^2 = 15\times7\times5\times3\left(\dfrac{k}{2}\right)^4$

$\therefore S = 15\sqrt{7}\left(\dfrac{k}{2}\right)^2$

$r = \dfrac{S}{s} = 15\sqrt{7}\left(\dfrac{k}{2}\right)^2 \div \dfrac{15}{2}k = \sqrt{7}\,\dfrac{k}{2}$

$R = \dfrac{abc}{4S} = \dfrac{4.5.6k^3}{4.15\sqrt{7}\,k^2/4} = \dfrac{8}{\sqrt{7}}k.$

$\therefore \dfrac{R}{r} = \dfrac{8}{\sqrt{7}} \div \dfrac{\sqrt{7}}{2} = \dfrac{16}{7}.$

60. (3) : $R\cos B = 2R\cos A\cos C \Rightarrow \tan A\tan C = 3$

61. (2) : $2\sin^2\theta - \cos2\theta = 0 \Rightarrow 1 - 2\cos2\theta = 0$

$\Rightarrow \cos2\theta = \dfrac{1}{2} \Rightarrow 2\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}, \dfrac{11\pi}{3}$

$\Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ ... (1)

where $\theta \in [0, 2\pi]$

Also $2\cos^2\theta - 3\sin\theta = 0 \Rightarrow 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow (2\sin\theta - 1)(\sin\theta + 2) = 0 \Rightarrow \sin\theta = \dfrac{1}{2}$

$[\because \sin\theta \ne -2]$

$\Rightarrow \theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ ... (2), where $\theta \in [0, 2\pi]$

Combining (1) and (2), we get $\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\therefore$ Two solutions are there.

62. (1) : In given ΔABC both $\dfrac{A}{2}$ and $\dfrac{B}{2}$ lie strictly between $\left(0, \dfrac{\pi}{2}\right)$ and $\sin x$ is always increasing in $\left(0, \dfrac{\pi}{2}\right)$ whereas $\cos x$ is always decreasing throughout $\left(0, \dfrac{\pi}{2}\right)$.

So, if $\dfrac{A}{2} > \dfrac{B}{2} \Rightarrow \sin\dfrac{A}{2} > \sin\dfrac{B}{2}$

or $x_1 > x_2$ and $\dfrac{1}{x_3} > \dfrac{1}{x_4}$

So, $x_1^{2007}x_4^{2006} = x_2^{2007}x_3^{2006}$ is not valid.

Similarly for $\frac{A}{2} < \frac{B}{2} \Rightarrow \sin\frac{A}{2} < \sin\frac{B}{2} \Rightarrow x_1 < x_2$

and $\frac{1}{x_3} < \frac{1}{x_4}$ again equality is not possible.

Therefore, only possible case is when $\frac{A}{2} = \frac{B}{2}$

$\Rightarrow x_1 = x_2$ and $\frac{1}{x_3} = \frac{1}{x_4}$

Hence, in that case ΔABC is isosceles with $\angle ABC = \angle CAB \Rightarrow BC = AC = 1$ unit

63. (2) : Use $\frac{1}{2} < -x < 1$

$\Rightarrow \cos^{-1}(-4x^3 + 3x) = \pi - 3\cos^{-1}x$

64. (5) : $(n-2)x^2 + 8x + n + 4 > 0 \ \forall \ x \in R$

$\Rightarrow n - 2 > 0 \Rightarrow n - 2 \geq 1$ is $n \geq 3$

and $D = 8^2 - 4(n-2)(n+4) < 0$

$\Rightarrow n > 4 \Rightarrow n \geq 5 \Rightarrow n = 5$

65. (2) : $\tan^{-1}x + \tan^{-1}(1/y) = \tan^{-1}(3)$

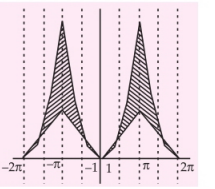
$\tan^{-1}(1/y) = \tan^{-1}(3) - \tan^{-1}x$

$\tan^{-1}(1/y) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$; $y = \frac{1+3x}{3-x}$

As x, y are positive integers, $x = 1, 2 \Rightarrow y = 2, 7$

Solution are $(x, y) = (1, 2)$ and $(2, 7)$

66. (22) : $f(x) = (\cos^{-1}(\cos x))^2$ and $g(x) = \cos^{-1}\cos x$



Required area

$$= 4\left[\int_0^1 (x - x^2)dx + \int_1^{\pi} (x^2 - x)dx\right]$$

$$= 4\left[\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^{\pi}\right] = \frac{4\pi^3 - 6\pi^2 + 4}{3}$$

67. (0) : $\sin^{-1}x + \cos^{-1}y + \sec^{-1}z \leq \frac{5\pi}{2}$

$t^2 - \sqrt{2\pi}t + 3\pi = \left(t - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \geq \frac{5\pi}{2}$

($\because$ left side is decreasing and right side is increasing. So they will match only on their common point of incidence.)

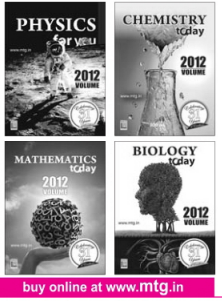
So solution exist if RHS = LHS $= \frac{5\pi}{2}$

i.e., $x = 1$, $y = -1$, $z = -1$, $t = \sqrt{\frac{\pi}{2}}$.

■■

AVAILABLE BOUND VOLUMES

of your favourite magazines



buy online at www.mtg.in

Volumes of the following years are available:

Physics For You

2012, 2011, 2010, 2008

Chemistry Today

2012, 2011, 2010, 2008

Mathematics Today

2012, 2011, 2010, 2008

Biology Today

2012, 2011, 2010, 2008

Price : ₹300 for each volume

POSTAGE FREE!

How to order : Send money by demand draft/money order. Demand Draft should be drawn in favour of **MTG Learning Media (P) Ltd**. Mention the volume you require along with your name and address.

Mail your order to :

Circulation Manager, MTG Learning Media (P) Ltd.
Plot 99, Sector 44 Institutional Area,
Gurgaon - 122 003 (HR)
Tel.: (0124) 4951200
E-mail : info@mtg.in Web : www.mtg.in

ENRICH YOUR CONCEPTS ON XI

Limits, Derivatives and Mathematical Reasoning

Based on NCERT Pattern

Series-9

LIMITS

Left Hand Limit : The left hand limit of $f(x)$ at $x = a$ is written as

$$\lim_{x \to a^-} f(x)$$

$x \to a^-$ means that x is tending to a from left hand side.

Right Hand Limit : The right hand limit of $f(x)$ at $x = a$ is written as

$$\lim_{x \to a^+} f(x)$$

$x \to a^+$ means that x is tending to a from the right hand side.

Limit of a Function at a Point : Limit of a function at a point is the value of the function at that point.

Remark 1 : For the existence of the limit of a function $f(x)$ at $x = a$, we must have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Remark 2 : If the left and right hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by

$$\lim_{x \to a} f(x).$$

Algebra of Limits : Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then

$$\lim_{x \to a}\left[f(x) + g(x)\right] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[f(x) - g(x)\right] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[f(x) \cdot g(x)\right] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ provided } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a}\left\{f(x)\right\}^{\{g(x)\}} = \left\{\lim_{x \to a} f(x)\right\}^{\lim_{x \to a} g(x)}$$

$$\lim_{x \to a} \lambda \cdot f(x) = \lambda \cdot \lim_{x \to a} f(x),\ \lambda \text{ being some real number.}$$

Limits of a Polynomial Function :

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are real nos. with $a_n \neq 0$. Then

$$\lim_{x \to a} f(x) = f(a)$$

Some Standard Limits :

1. $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$, n being a positive integer
2. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
3. $\lim_{x \to a} \frac{\sin(x - a)}{x - a} = 1$
4. $\lim_{x \to 0} \cos x = 1$
5. $\lim_{x \to 0} \frac{\tan x}{x} = 1$
6. $\lim_{x \to a} \frac{\tan(x - a)}{x - a} = 1$
7. $\lim_{x \to 0} \frac{x}{\tan x} = 1$
8. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
9. $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$
10. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 1$
11. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
12. $\lim_{x \to 0} (1 + x)^{1/x} = e$

DERIVATIVES

Definition : Let $f(x)$ be a real valued function and a be a point in its domain of definition. The derivative of $f(x)$ at a is defined as

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

provided this limit exists.

MTG

36 YEARS CHAPTERWISE SOLUTIONS

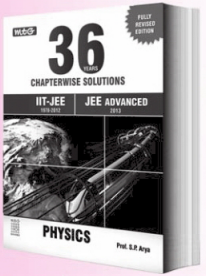
Is there a fool-proof method to guarantee your success in JEE ADVANCED - 2014?

It is time to give you a clear perspective of the relationship between preparing for the exam and succeeding at it.

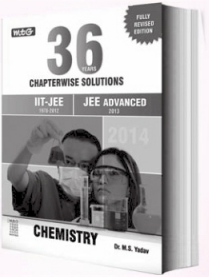
According to the findings of our panel of experts, apart from starting as early as 2 years in advance and putting in 3 to 5 hours everyday, successful students devote 2 years before the exams getting familiar with the problems which have appeared in the last JEEs. This is the secret of their success.

Our Key For Your Success

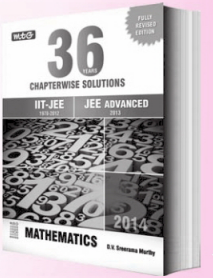
It is in the last area, that we can really help you. We have categorized chapterwise problems and solutions from 36 years papers of JEE engaging some of the finest minds in the country. The result of this laborious exercise has emerged as the **MTG 36 years Chapterwise Solutions.**



₹350



₹350



₹350

A Must for Every IIT Hopeful

Without doubt the **36 years Chapterwise Solutions** is the single most powerful tool available to a serious student.

It is a tool designed to fine tune for all the efforts that you have put in until now and our meticulously demonstrated solutions will clarify your doubt beyond doubt.

So go ahead. Lay your hands on the most comprehensive success tool available.

Get hold of the **MTG 36 years Chapterwise Solutions** just when it matters.

Buy online at www.mtg.in

Available at leading bookshops throughout India.

MTG BOOKS

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

Derivative of $f(x)$ at a is denoted by $f'(a)$.

$\therefore \quad f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

The First Principle of Derivative : Suppose f is a real valued function, the function defined by

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f at x and is denoted by $f'(x)$. Thus,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Note : $f'(x)$ is also denoted by $\frac{d}{dx}[f(x)]$ or $D[f(x)]$ or $\frac{dy}{dx}$ where $y = f(x)$.

Algebra of Derivative of Functions : Let $f(x)$ and $g(x)$ be two functions such that their derivatives are defined in a common domain. Then

Derivative of sum of two functions	$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Derivative of difference of two functions	$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
Product Rule	$\frac{d}{dx}[f(x)\cdot g(x)]$ $= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ $= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$

Derivative of a Polynomial :

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ be a polynomial function in x, where a_i's are real numbers and $a_n \neq 0$. Then

$$\frac{d}{dx}f(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \ldots + 2a_2 x + a_1$$

Some Standard Derivatives :

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(a^x) = a^x \log_e a$
4. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$
5. $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$, $a > 0, a \neq 1, x > 0$.
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$
8. $\frac{d}{dx}(\tan x) = \sec^2 x$
9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
10. $\frac{d}{dx}(\sec x) = \sec x \tan x$
11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Note : $\frac{d}{dx}[c\cdot f(x)] = c\frac{d}{dx}f(x)$, c being a constant.

MATHEMATICAL REASONING

Statement : A sentence is called a mathematically acceptable statement or simply a statement if it is either true or false but not both.

Negation of a Statement : The denial of a statement p is called its negation and is written as $\sim p$, and read as 'not p'.

Compound Statements :

1. A statement is called a compound statement if it is made up of two or more simple statements. The simple statements are called component statements of the compound statement.
2. Compound statements are obtained by using connecting words like "and", "or" etc and phrases like "If-then", "Only if", "if and only if", "There exists", "For all" etc.
3. The compound statement with "AND" is
   (i) true if all its component statements are true.
   (ii) false if any of its component statements is false.
4. The compound statement with "OR" is
   (i) true when one component statement is true or both the component statements are true.
   (ii) false when both the component statements are false.
5. A statement is said to be valid or invalid according as it is true or false.

Quantifiers : "There exists" and "For every". These two phrases are called quantifiers. Depending upon the context the phrase "There exists" can also be replaced

by the equivalent phrases "There is" or "There is at least one" or "It is possible to find" or "some".

Implications : Statements of the form "if p then q", "p only if q" and "if and only if" are called implications.

A sentence with "if p then q" can be written in the following ways :

(i) p implies q (denoted by $p \Rightarrow q$)
(ii) p is sufficient condition for q
(iii) q is necessary condition for p
(iv) p only if q
(v) $\sim q$ implies $\sim p$

Converse : If p and q are two statements, then the converse of the implication "if p then q" is "if q then p".

Inverse : If p and q are two statements, then the inverse of "If p then q" is "If $\sim p$ then $\sim q$."

If and only if Implication : If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.

If p and q are two mathematical statements, then the statement :

(i) "If p then q" is true if
  (a) p is true $\Rightarrow q$ is true
  or
  (b) q is false $\Rightarrow p$ is false
  or
  (c) p is true and q is false leads us to a contradiction.

(ii) "p if and only if q" is true, if
  (a) p is true $\Rightarrow q$ is true and
  (b) q is true $\Rightarrow p$ is true.

Note :

1. The contrapositive of the statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.
2. The converse of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
3. The inverse of the statement $p \Rightarrow q$ is the statement $\sim p \Rightarrow \sim q$.

VERY SHORT ANSWER TYPE

1. Evaluate $\lim\limits_{x\to 3}\left(\dfrac{x^2+9}{x+3}\right)$.
2. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right)$.
3. Find the derivative of $\sin x$ at $x = 0$.
4. If $f(x) = x^n$ and if $f'(1) = 10$, find the value of n.
5. If $y = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$, show that $\dfrac{dy}{dx} = y$.
6. Write the value of the derivative of $f(x) = |x-1| + |x-3|$ at $x = 2$.
7. Write the negation of each of the following statements.
   (i) 3 is greater than 5.
   (ii) All the sides of an equilateral triangle are of the same length.
8. Which of the following are statements?
   (i) $6 + x = 0$.
   (ii) How black is the dog!
9. Given below are two statements:
   p : 80 is a multiple of 5.
   q : 80 is a multiple of 4.
   Write the compound statement connecting these two statements with "and" and check its validity.

SHORT ANSWER TYPE

10. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{3^{2x}-1}{2^{3x}-1}\right)$.
11. Evaluate $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}$.
12. Evaluate $\lim\limits_{x\to a}\left(\dfrac{x^m-a^m}{x^n-a^n}\right)$.
13. Evaluate $\lim\limits_{x\to \pi}\left(\dfrac{1+\cos x}{\tan^2 x}\right)$
14. Differentiate the function $\sin^2 x$ with respect to x from the first principle.
15. Differentiate the function $\dfrac{\sec x - 1}{\sec x + 1}$ w.r.t. x.
16. Differentiate the function $(ax^2 + \sin x)(p + q\cos x)$ with respect to x.
17. Show that the statement p : "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by method of contradiction.
18. Check the validity of the statement given below by contradiction method.
    "p : the sum of an irrational number and a rational number is irrational."

LONG ANSWER TYPE

19. Evaluate $\lim\limits_{x\to 0}\dfrac{\left(1-\cos x\sqrt{\cos 2x}\right)}{x^2}$.
20. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}\right)$.
21. Differentiate the function $\sin\sqrt{x}$ with respect to x from first principle.
22. Differentiate $x^2 \cos x$ w.r.t. x from first principle.

23. Show that the following statement is true by the method of contrapositive.
p : If x is an integer and x^2 is even, then x is also even.

SOLUTIONS

1. $\lim\limits_{x\to 3}\dfrac{x^2+9}{x+3}=\dfrac{3^2+9}{3+3}=\dfrac{18}{6}=3$

2. $\lim\limits_{x\to 0}\left(\dfrac{e^{3x}-1}{x}\right)=\lim\limits_{3x\to 0}\left\{\left(\dfrac{e^{3x}-1}{3x}\right)\times 3\right\}$

$[\because x\to 0\Rightarrow 3x\to 0]$

$=3\times\lim\limits_{y\to 0}\left(\dfrac{e^{y}-1}{y}\right)$, where $y=3x$

$=(3\times 1)=3 \quad \left[\because \lim\limits_{y\to 0}\left(\dfrac{e^{y}-1}{y}\right)=1\right]$.

3. Let $f(x)=\sin x$. Then

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h}$$
$$=\lim_{h\to 0}\frac{\sin h}{h}=1$$

4. We have, $f(x)=x^n$,
Differentiating both sides w.r.t. x,
we get $f'(x)=nx^{n-1}$.
Putting $x=1$, we get
$f'(1)=n\Rightarrow 10=n$.

5. We have, $y=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots$

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(1)+\frac{d}{dx}\left(\frac{x}{1!}\right)+\frac{d}{dx}\left(\frac{x^2}{2!}\right)+\frac{d}{dx}\left(\frac{x^3}{3!}\right)+\ldots$$

$$\text{or, } \frac{dy}{dx}=\frac{d}{dx}(1)+\frac{1}{1!}\frac{d}{dx}(x)+\frac{1}{2!}\frac{d}{dx}(x^2)+\frac{1}{3!}\frac{d}{dx}(x^3)+\ldots$$

$$\text{or, } \frac{dy}{dx}=0+1+\frac{1}{2!}(2x)+\frac{1}{3!}(3x^2)+\ldots$$

$$\text{or, } \frac{dy}{dx}=1+x+\frac{x^2}{2!}+\ldots$$

$$\text{or, } \frac{dy}{dx}=y.$$

6. Given : $f(x)=|x-1|+|x-3|$
At $x=2$, $|x-1|=x-1$ and $|x-3|=-(x-3)$
$\therefore\quad f(x)=x-1-(x-3)=2$
$\Rightarrow\quad \dfrac{d}{dx}f(x)=\dfrac{d}{dx}(2)=0$

$\therefore\quad \left.\dfrac{d}{dx}f(x)\right|_{x=2}=0.$

7. (i) Let p : 3 is greater than 5.
$\sim p$: 3 is not greater than 5 or $\sim p$: $3\le 5$.
(ii) Let p : All the sides of an equilateral triangle are of the same length.
$\sim p$: Not all the sides of an equilateral triangle are of the same length.

8. (i) Without knowing the value of x, we do not know whether the given sentence is true or false. So, it is a sentence but not a statement.
(ii) Being an exclamatory sentence, it is not a statement.

9. The compound statement is
"80 is multiple of 5 and 4."
We know that 80 is a multiple of 5 as well as 4. So, p and q are true statements.
Hence, the compound statement is also true *i.e.*, the compound statement "p and q" is a valid statement.

10. $\lim\limits_{x\to 0}\left(\dfrac{3^{2x}-1}{2^{3x}-1}\right)=\lim\limits_{x\to 0}\dfrac{\left\{\left(\dfrac{3^{2x}-1}{2x}\right)\cdot 2x\right\}}{\left\{\left(\dfrac{2^{3x}-1}{3x}\right)\cdot 3x\right\}}$

$$=\frac{2}{3}\cdot\frac{\lim\limits_{2x\to 0}\left(\dfrac{3^{2x}-1}{2x}\right)}{\lim\limits_{3x\to 0}\left(\dfrac{2^{3x}-1}{2x}\right)}$$

$$=\frac{2}{3}\cdot\frac{\log 3}{\log 2}\quad\left[\because \lim_{y\to 0}\left(\frac{a^y-1}{y}\right)=\log a\right]$$

$$=\frac{2\log 3}{3\log 2}=\frac{\log(3)^2}{\log(2^3)}=\frac{\log 9}{\log 8}.$$

11. $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x^2+x-3)}=\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$$=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$
$$=\lim_{x\to 1}\frac{(2x-3)(x-1)}{(2x+3)(x-1)(\sqrt{x}+1)}$$
$$=\lim_{x\to 1}\frac{(2x-3)}{(2x+3)(\sqrt{x}+1)}$$
$$=\frac{(2\times 1-3)}{(2\times 1+3)(\sqrt{1}+1)}=\frac{-1}{10}$$

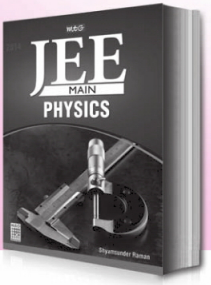
12. $\lim\limits_{x\to a}\left(\dfrac{x^m-a^m}{x^n-a^n}\right)=\lim\limits_{x\to a}\left\{\left(\dfrac{x^m-a^m}{x-a}\right)\div\left(\dfrac{x^n-a^n}{x-a}\right)\right\}$

$$=\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)\div\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$$
$$=(ma^{m-1})\div(na^{n-1})$$

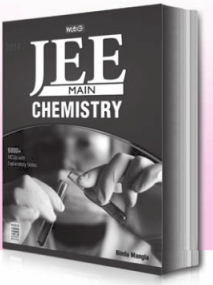
mtG

BEST TOOLS FOR SUCCESS IN

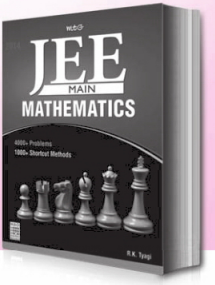
JEE Main



₹575



₹650



₹550

- Based on latest pattern of JEE Main
- Illustrations and MCQs to clear your concepts
- Covers complete syllabus
- Also contains previous years' MCQs of AIEEE, JEE Main and other Engineering Entrance Exams.

Success in JEE 2014 Now Made Easy !

Buy online at www.mtg.in

Available at leading book-shops throughout India.

mtG BOOKS

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

$$= \frac{ma^{m-1}}{na^{n-1}} = \frac{m}{n}a^{m-n}.$$

13. $$\lim_{x\to\pi}\left(\frac{1+\cos x}{\tan^2 x}\right) = \lim_{x\to\pi}\frac{\cos^2 x(1+\cos x)}{\sin^2 x}$$

$$= \lim_{x\to\pi}\frac{\cos^2 x(1+\cos x)}{(1-\cos^2 x)}$$

$$= \lim_{x\to\pi}\frac{\cos^2 x}{(1-\cos x)} = \frac{1}{2} \text{ [putting } x = \pi\text{].}$$

14. Let $f(x) = \sin^2 2x$. Then, $f(x+h) = \sin^2(x+h)$

$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin^2(x+h)-\sin^2 x}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin(x+h+x)\sin(x+h-x)}{h}$

$[\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)]$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\sin(2x+h)$

$= 1\cdot(\sin 2x) = \sin 2x$

$\therefore \quad \frac{d}{dx}(\sin^2 x) = \sin 2x$

15. We have, $f(x) = \left(\frac{\sec x - 1}{\sec x + 1}\right)$

$$\frac{d}{dx}(f(x)) = \frac{(\sec x+1)\cdot\frac{d}{dx}(\sec x-1) - (\sec x-1)\cdot\frac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$= \frac{(\sec x+1)(\sec x\tan x - 0) - (\sec x-1)(\sec x\tan x+0)}{(\sec x+1)^2}$$

$$= \frac{\sec^2 x\tan x + \sec x\tan x - \sec^2 x\tan x + \sec x\tan x}{(\sec x+1)^2}$$

$$= \frac{2\sec x\tan x}{(\sec x+1)^2} = \frac{2\sin x}{(1+\cos x)^2}$$

16. We have, $f(x) = \{(ax^2 + \sin x)(p + q\cos x)\}$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(ax^2+\sin x)\times(p+q\cos x) + (ax^2+\sin x)\frac{d}{dx}(p+q\cos x)$$

$$= \left\{\frac{d}{dx}(ax^2)+\frac{d}{dx}(\sin x)\right\}(p+q\cos x) + (ax^2+\sin x)\left\{\frac{d}{dx}(p)+\frac{d}{dx}(q\cos x)\right\}$$

$$= (a(2x)+\cos x)(p+q\cos x) + (ax^2+\sin x)(0-q\sin x)$$

$$= (2ax+\cos x)(p+q\cos x) - (ax^2+\sin x)\,q\sin x$$

17. Method of contradiction : If possible, let p be not true. Then,

p is not true

$\Rightarrow$ $\sim p$ is true

$\Rightarrow$ $\sim(q \Rightarrow r)$ is true $\quad [\because p : q \Rightarrow r]$

$\Rightarrow$ q and $\sim r$ is true $\quad [\because \sim(q \Rightarrow r) \equiv q \text{ and } \sim r]$

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$ and $x \neq 0$

$\Rightarrow$ $x = 0$ and $x \neq 0$

This is a contradiction.

Hence, p is true.

18. If possible, let p be not true. Then,

p is false

$\Rightarrow$ The sum of an irrational number and a rational number is not irrational.

$\Rightarrow$ There exists an irrational number x (say) and a rational number y (say) such that $x + y$ is not irrational.

$\Rightarrow$ $x + y$ is rational say, z

$\Rightarrow$ $x + y = z$

$\Rightarrow$ $x = z - y$

$\Rightarrow$ x is rational $\quad [\because z - y \text{ is rational}]$

But, x is irrational. So, we arrive at a contradiction.

Thus, our supposition is wrong.

Hence, p is true.

19. $$\lim_{x\to 0}\frac{(1-\cos x\sqrt{\cos 2x})}{x^2}$$

$$= \lim_{x\to 0}\left\{\frac{(1-\cos x\sqrt{\cos 2x})}{x^2}\times\frac{(1+\cos x\sqrt{\cos 2x})}{(1+\cos x\sqrt{\cos 2x})}\right\}$$

$$= \lim_{x\to 0}\left\{\frac{(1-\cos^2 x\cos 2x)}{x^2}\times\frac{1}{(1+\cos x\sqrt{\cos 2x})}\right\}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x\cos 2x)}{x^2}\times\lim_{x\to 0}\frac{1}{(1+\cos x\sqrt{\cos 2x})}$$

$$= \lim_{x\to 0}\left\{\frac{1-\cos^2 x(1-2\sin^2 x)}{x^2}\right\}\times\frac{1}{(1+1\sqrt{1})}$$

$$= \lim_{x\to 0}\frac{(1-\cos^2 x+2\sin^2 x\cos^2 x)}{x^2}\times\frac{1}{2}$$

$$= \frac{1}{2}\times\lim_{x\to 0}\frac{(\sin^2 x+2\sin^2 x\cos^2 x)}{x^2}$$

$$= \frac{1}{2}\times\lim_{x\to 0}\frac{\sin^2 x(1+2\cos^2 x)}{x^2}$$

$$= \frac{1}{2}\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\times\lim_{x\to 0}(1+2\cos^2 x)$$

$$= \frac{1}{2}\times 1^2\times(1+2\times 1^2) = \frac{3}{2}.$$

20. $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x+\sin 6x}{\sin 5x-\sin 3x}\right)$

$=\lim\limits_{x\to 0}\left(\dfrac{2\sin 4x\cos 2x}{2\cos 4x\sin x}\right)$

$=\lim\limits_{x\to 0}\left(\dfrac{\sin 4x\cos 2x}{\cos 4x\sin x}\right)$

$=\lim\limits_{x\to 0}\left\{\dfrac{\sin 4x}{4x}\times\dfrac{x}{\sin x}\times\cos 2x\times\dfrac{1}{\cos 4x}\times 4\right\}$

$=4\times\lim\limits_{4x\to 0}\dfrac{\sin 4x}{4x}\times\lim\limits_{x\to 0}\left(\dfrac{x}{\sin x}\right)\times$

$\lim\limits_{2x\to 0}\cos 2x\times\dfrac{1}{\lim\limits_{4x\to 0}\cos 4x}$

(If $x\to 0$, then $2x\to 0$ and $4x\to 0$)

$=\left(4\times 1\times 1\times 1\times\dfrac{1}{1}\right)=4.$

21. Let $f(x)=\sin\sqrt{x}$. Then, $f(x+h)=\sin\sqrt{x+h}$

$\therefore\quad \dfrac{d}{dx}\big(f(x)\big)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow\quad \dfrac{d}{dx}\big(f(x)\big)=\lim\limits_{h\to 0}\dfrac{\sin\sqrt{x+h}-\sin\sqrt{x}}{h}$

$=\lim\limits_{h\to 0}\dfrac{2\sin\left(\dfrac{\sqrt{x+h}-\sqrt{x}}{2}\right)\cos\left(\dfrac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\dfrac{\sqrt{x+h}-\sqrt{x}}{2}\right)}\cdot$

$\dfrac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)h}\cos\left(\dfrac{\sqrt{x+h}+\sqrt{x}}{2}\right)$

$=\lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\dfrac{\sqrt{x+h}-\sqrt{x}}{2}\right)}\cdot\lim\limits_{h\to 0}\dfrac{x+h-x}{\left(\sqrt{x+h}+\sqrt{x}\right)h}\cdot$

$\lim\limits_{h\to 0}\cos\left(\dfrac{\sqrt{x+h}+\sqrt{x}}{2}\right)$

$\Rightarrow\quad \dfrac{d}{dx}\big(f(x)\big)=1\times\lim\limits_{h\to 0}\dfrac{1}{\sqrt{x+h}+\sqrt{x}}\times$

$\lim\limits_{h\to 0}\cos\left(\dfrac{\sqrt{x+h}+\sqrt{x}}{2}\right)$

$\Rightarrow\quad \dfrac{d}{dx}\big(f(x)\big)=1\times\dfrac{1}{2\sqrt{x}}\times\big(\cos\sqrt{x}\big)$

$\therefore\quad \dfrac{d}{dx}\big(\sin\sqrt{x}\big)=\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$

22. Let $f(x)=x^2\cos x$. Then, $f(x+h)=(x+h)^2\cos(x+h)$

$\dfrac{d}{dx}\big(f(x)\big)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow\quad \dfrac{d}{dx}\big(f(x)\big)=\lim\limits_{h\to 0}\dfrac{(x+h)^2\cos(x+h)-x^2\cos x}{h}$

$=\lim\limits_{h\to 0}\dfrac{\left(x^2+2hx+h^2\right)\cos(x+h)-x^2\cos x}{h}$

$=\lim\limits_{h\to 0}\dfrac{\left\{x^2\cos(x+h)-x^2\cos x\right\}+2hx\cos(x+h)+h^2\cos(x+h)}{h}$

$=\lim\limits_{h\to 0}\left[x^2\left\{\dfrac{\cos(x+h)-\cos x}{h}\right\}+2x\cos(x+h)+h\cos(x+h)\right]$

$=\lim\limits_{h\to 0}(-2x^2)\dfrac{\sin\left(x+\dfrac{h}{2}\right)\cdot\sin\dfrac{h}{2}}{h}+\lim\limits_{h\to 0}2x\cos(x+h)$

$+\lim\limits_{h\to 0}h\cos(x+h)$

$=(-x^2)\lim\limits_{h\to 0}\dfrac{\sin\left(x+\dfrac{h}{2}\right)\cdot\sin\dfrac{h}{2}}{\dfrac{h}{2}}+\lim\limits_{h\to 0}2x\cos(x+h)$

$+\lim\limits_{h\to 0}h\cos(x+h)$

$=(-x^2)\sin x+2x\cos x$

$\Rightarrow\quad \dfrac{d}{dx}\big(f(x)\big)=-x^2\sin x+2x\cos x.$

23. Let q and r be the statements given by

q : x is an integer and x^2 is even

r : x is an even integer.

Then, p : "If q then r".

If possible, let r be false. Then.

r is false

$\Rightarrow$ x is not an even integer

$\Rightarrow$ x is an odd integer

$\Rightarrow$ $x=(2n+1)$ for some integer n

$\Rightarrow$ $x^2=4n^2+4n+1$

$\Rightarrow$ $x^2=4n\,(n+1)+1$

$\Rightarrow$ x^2 is an odd integer [$\because 4n\,(n+1)$ is even]

$\Rightarrow$ q is false.

Thus, r is false $\Rightarrow q$ is false.

Hence, p : "If q, then r" is a true statement.

■■

2014 CBSE-BOARD CHAPTERWISE PRACTICE PAPER

Series-9

Linear Programming and Probability

LINEAR PROGRAMMING

- **Definition :** Linear programming is a method for finding the optimal values (maximum or minimum) of quantities subject to the constraints when relationship is expressed as linear equations or inequalities.

Terms used in LPP

- **Objective Function :** The linear function which is to be maximised or minimised under given constraints.
- **Constraints :** The restrictions or linear inequalities or equations in the variables of an LPP, which describe the conditions under which the optimisation is to be accomplished.
- **Non-negativity Constraints :** The assumption that negative values of variables are not possible in the solution. They are described as $x \geq 0$, $y \geq 0$.
- **Feasible Solution :** The set of points, whose co-ordinates satisfy the constraints and non-negativity of a linear programming problem, is said to be the feasible solution.
- **Feasible Region :** The common region determined by all the constraints of an LPP is called the feasible region and every point in this region is a feasible solution of given LPP.
- **Optimal Feasible Solution :** Any point which maximises or minimises the objective function.
- **Optimal Value of Objective Function :** The optimal value (maximum or minimum) of the objective function is obtained at a vertex of the feasible region (if it is bounded). If there are more than one points (vertices) where the objective function is optimum (maximum or minimum), then every point on the line segment joining any two such vertices optimizes the objective function.
- **Corner Point :** A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

- **Some Important Theorems of LPP**

(i) **Theorem 1 :** Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

(ii) **Theorem 2 :** Let R be the feasible region for a linear programming problem, and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at a corner point (vertex) of R.

Remark : If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it must occur at a corner point of R. (By Theorem 1)

Step, How to draw the graph of inequalities.

First step : Plot the graph of the inequalities describing the various constraints (structural) on the graph paper.

Second step : Find the portion of the graph paper in the first quadrant ($\because$ of non-negativity constraints) which is common to the graphs plotted in the first step. Locate the extreme points (i.e., corner points) of this region (known as feasible region).

Third step : Find the value of the objective function corresponding to each corner point. The points which corresponds to the optimum (*i.e.*, maximum or minimum) value of the objective function is (are) required solution (s) of the given linear programming problem.

PROBABILITY

Conditional Probability : If E_1 and E_2 are any two events, then probability of E_2 when E_1 has already occurred is given by $P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$, provided $P(E_1) \neq 0$

Properties of Conditional Probability :

(i) If E_1 and E_2 are any two events associated with an experiment and $P(E_2) \neq 0$, then $0 \leq P(E_1/E_2) \leq 1$.

(ii) If E is any event associated with an experiment, then $P(E/E) = P(S/E) = 1$, S being the sample space.

(iii) If E, E_1 and E_2 are events associated with an experiment and $P(E) \neq 0$, then
$P((E_1 \cup E_2)/E) = P(E_1/E) + P(E_2/E) - P((E_1 \cap E_2)/E)$.
In particular, if E_1 and E_2 are mutually exclusive, then
$P((E_1 \cup E_2)/E) = P(E_1/E) + P(E_2/E)$.

(iv) If E_1 and E_2 are any two events associated with an experiment, then
$P(E_1^C/E_2) = 1 - P(E_1/E_2)$.

Multiplication Theorem on Probability :

(i) For two events : If E_1 and E_2 are two events then
$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2/E_1) = P(E_2) \cdot P(E_1/E_2)$
where $P(E_1) \neq 0$ and $P(E_2) \neq 0$.

(ii) For three events : If E_1, E_2 and E_3 are three events, then
$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2/E_1) \cdot P(E_3/E_1 \cap E_2)$

(iii) For n events
$P(E_1 \cap E_2 \cap ... \cap E_n) = P(E_1) \cdot P(E_2/E_1) \cdot P(E_3/E_1 \cap E_2) ... P(E_n/E_1 \cap E_2 \cap ... \cap E_n)$

Independent Events :

(i) Two events E_1 and E_2 are called independent if $P(E_1/E_2) = P(E_1)$, $P(E_2) \neq 0$ and $P(E_2/E_1) = P(E_2)$, $P(E_1) \neq 0$.

(ii) For independent events E_1 and E_2, $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$

Theorem of Total Probability : If E_1, E_2, E_3, ..., E_n be a set of mutually exclusive and exhaustive events associated with a random experiment and E be any event associated with the same experiment, then

$$P(E) = \sum_{i=1}^{n} P(E/E_i)P(E_i)$$

Bayes' Theorem : If E_1, E_2, E_3, ..., E_n form a set of mutually exclusive and exhaustive events and E is any event associated with the same experiment, then

$$P(E_i/E) = \frac{P(E/E_i)P(E_i)}{\sum_{i=1}^{n} P(E/E_i)P(E_i)}$$

Random Variable : A random variable is a real valued function X defined on the sample space S of a random experiment which associates a unique real number $X(\omega)$ to each $\omega \in S$.
So, Domain of $X = S$; Range of $X = \{X(\omega) : \omega \in S\}$.
A random variable is said to be discrete if it can assume only finite or countably infinite number of values and it is said to be continuous if it can assume any real value.

(i) The sample spaces of the associated random experiments are different *i.e.* the domains of the two variables are different.

(ii) The sets of possible values of the two random experiments are different *i.e.* the ranges of the two variables are different.

(iii) The associated sample spaces and the set of possible values are the same but the two variables take different values at least at one point of the common sample space (domain).

Probability Distribution of Random Variable : In general if $p_1, p_2, p_3, ..., p_n$ are the respective probabilities of the values $x_1, x_2, ..., x_n$ of a random variable X, then $p_1 + p_2 + p_3 + ... + p_n = 1$ and the table is called the probability distribution of X.

X	x_1	x_2	x_3	...	x_n
$P(X)$	p_1	p_2	p_3	...	p_n

Thus, we see that the different values $x_1, x_2, x_3, ..., x_n$ of X together with their corresponding probabilities form a probability distribution of the random variable X.

Mean of a Random Variable : The mean of a random variable X is also called expectation of X, denoted

$$E(X) = \mu = x_1p_1 + x_2p_2 + ... + x_np_n = \sum_{i=1}^{n} x_ip_i$$

Variance of a Random Variable : Let X be a random variable whose possible values $x_1, x_2, ..., x_n$ occur with probabilities $p(x_1)$, $p(x_2)$, ..., $p(x_n)$ respectively,

$$\sigma_X^2 = V(X) = \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i\right)^2$$

Standard Deviation : $\sigma_X = \sqrt{V(X)}$

Bernoulli Trials : Trials of a random experiment are Bernoulli trials, if

(i) They are finite.

(ii) They are independent of each other.

(iii) Each trial has two outcomes : success or failure.

(iv) Probability of success or failure remain the same in each trial.

Binomial Distribution : A random variable X which takes values 0, 1, 2, ..., n follows binomial distribution if its probability distribution function is given by $P(X = r) = {}^nC_r p^r q^{n-r}$, $r = 0, 1, 2, ..., n$, where $p + q = 1$, $p, q > 0$, p = probability of success, q = probability of failure, n = number of trials.

Mean, Variance and Standard Deviation of Binomial Distribution : Mean = np, Variance = npq and Standard Deviation = $\sqrt{npq}$

Recurrence Relation : If $P(0)$ is known, we can easily find $P(1)$, $P(2)$ … etc. by applying the following recurrence formula $P(r+1)=\left(\frac{n-r}{r+1}\right)\frac{p}{q}P(r)$.

VERY SHORT ANSWER TYPE (1 MARK)

1. An airline agrees to charter planes for a group. The group needs at least 160 first class seats and at least 300 tourist class seats. The airline must use at least two of its model 314 planes which have 20 first class and 30 tourist class seats and some of its model 535 planes which have 20 first class seats and 60 tourist class seats. Each flight of a model 314 plane costs ₹ 1 lakh and each flight of a model 535 plane costs ₹ 1.5 lakh. Formulate this as an LPP to minimize the flight cost.

2. The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both?

3. Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value x has the following form, where k is some unknown constant.

$$P(X=x)=\begin{cases}0.1 & ,\text{if } x=0\\ kx & ,\text{if } x=1 \text{or} 2\\ k(5-x) & ,\text{if } x=3 \text{or} 4\\ 0 & ,\text{otherwise}\end{cases}$$

Find the value of k.

4. Prove that the mean of a binomial distribution is always greater than the variance.

5. A four digit number is formed using the digits 1, 2, 3, 5 with no repetitions. Write the probability that the number is divisible by 5.

SHORT ANSWER TYPE (4 MARKS)

6. Solve the following linear programming problem graphically :
Maximize $Z = 50x + 15y$
Subject to
$5x + y \le 100$
$x + y \le 60$
$x, y \ge 0$.

7. If the sum of the mean and variance of a binomial distribution for 5 trials is 1.8, find the distribution.

8. Solve the following LPP by graphical method:
Minimize $Z = 20x + 10y$
Subject to $x + 2y \le 40$
$3x + y \ge 30$
$4x + 3y \ge 60$
and $x, y \ge 0$

9. Mother, father and son line up at random for a family picture. Find $P(A/B)$, if A and B are defined as follows :
A = Son on one end, B = Father in the middle

10. A coin is tossed until a head appears or the tail appears 4 times in succession. Find the probability distribution of the number of tosses.

11. In a hurdles race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

LONG ANSWER TYPE (6 MARKS)

12. Find the mean, variance and standard deviation of the number of heads in a simultaneous toss of three coins.

13. Consider the experiment of tossing a coin. If the coin shows head toss it again but if it shows tail then throw a die. Find the conditional probability of the event, 'the die shows a number greater than 4, given that 'there is at least one tail'.

14. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copied it, is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.

15. A company has factories located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 7, 6 and 4 units of the commodity while the weekly production capacities of the factories at P and Q are respectively 9 and 8 units. The cost of transportation per unit is given below.

From \ To	Cost (in ₹)		
	A	B	C
P	16	10	15
Q	10	12	10

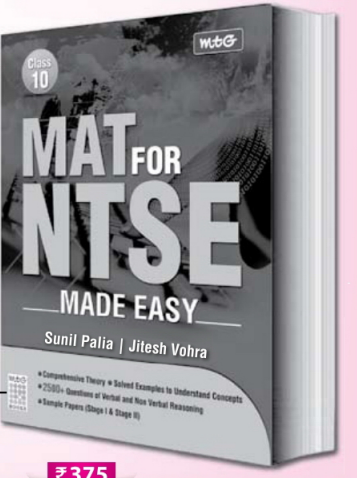
How many units should be transported from each factory to each depot in order that the

mtG

Attention Class 10 STUDENTS!

The most Up-to-Date and Authentic Study Book for National Talent Search Examination

MAT FOR NTSE
MADE EASY



Now on Sale !!!

- Verbal Reasoning
- Non-Verbal Reasoning
- Model Test Papers

₹375

Buy online at **www.mtg.in**

Available at all leading bookshops throughout India

mtG BOOKS

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

transportation cost is minimum? Formulate the above LPP mathematically and then solve it.

16. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760.00 to invest and has space for at most 20 items. A fan costs him ₹ 360.00 and a sewing machine ₹ 240.00. His expectation is that he can sell a fan at a profit of ₹ 22.00 and a sewing machine at a profit of ₹ 18.00. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit? Translate this problem mathematically and then solve it.

SOLUTIONS

1. Let the number of model 314 planes be x and that of model 535 planes be y. Then

Minimize $Z = x + 1.5\,y$

subject to constraints :

$20x + 20y \geq 160 \Rightarrow x + y \geq 8$

$30x + 60y \geq 300 \Rightarrow x + 2y \geq 10$

and $x \geq 0,\ y > 0.$

2. Consider the following events :

A = Person gets an electric contract,

B = Person gets plumbing contract

We have,

$$P(A) = \frac{2}{5}, P(\overline{B}) = \frac{4}{7} \text{ and } P(A \cup B) = \frac{2}{3}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad \frac{2}{3} = \frac{2}{5} + \left(1 - \frac{4}{7}\right) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$$

3. The probability distribution of X is

x :	0	1	2	3	4
$P(x)$:	0.1	k	$2k$	$2k$	k

$\therefore \quad P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = 1$

$\Rightarrow \quad 0.1 + k + 2k + 2k + k = 1$

$\Rightarrow \quad 6k = 0.9$

$\Rightarrow \quad k = 0.15$

4. Let X be a binomial variate with parameters n and p. Then,

Mean = np and Variance = npq

$\therefore \quad$ Mean – Variance = $np - npq = np\,(1 - q) = np^2$

$\Rightarrow \quad$ Mean – Variance > 0

$[\because\ n \in N \text{ and } p > 0 \ \ \therefore\ np^2 > 0]$

$\Rightarrow \quad$ Mean > Variance.

5. A four digit number formed using the digits 1, 2, 3 and 5 without repetitions will be divisible by 5 if and only if the unit's digit is 5.

So, the no. of ways to fill the rest three places is 3!.

Further the total no. of nos. that can be formed = 4!

$\therefore \quad$ Required probability = $\dfrac{3!}{4!} = \dfrac{1}{4}$.

6. We first convert the inequations into equations to obtain the lines $l_1 : 5x + y = 100$, $l_2 : x + y = 60$, $x = 0$ and $y = 0$.

The line l_1 meets the coordinate axes at $A_1(20, 0)$ and $B_1(0, 100)$.

Clearly (0, 0) satisfies the inequation $5x + y \leq 100$. So, the region in xy-plane that contains the origin represents the solution set of the given equation.

The line l_2 meets the coordinate axes at $A_2(60, 0)$ and $B_2(0, 60)$.

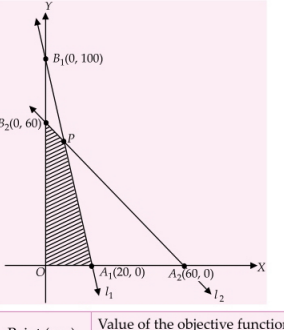
Clearly (0, 0) satisfies the inequation $x + y \leq 60°$. So, the region in xy-plane that contains the origin represents the solution set of given inequation.

Also, $x = 0$ is the y-axis and $y = 0$ is the x-axis.

The feasible region of the LPP is shaded in figure.

The coordinate of the corner-points of the feasible region OA_1PB_2 are $O(0, 0)$, $A_1(20, 0)$, $P(10, 50)$ and $B_2(0, 60)$.

The value of the objective function at these points are given in the following table :



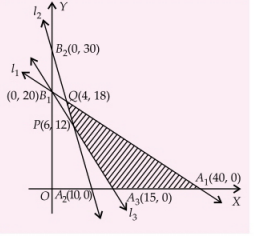
Point (x, y)	Value of the objective function $Z = 50x + 15y$
$O(0, 0)$	$Z = 50 \times 0 + 15 \times 0 = 0$
$A_1(20, 0)$	$Z = 50 \times 20 + 15 \times 0 = 1000$
$P(10, 50)$	$Z = 50 \times 10 + 15 \times 50 = 1250$
$B_2(0, 60)$	$Z = 50 \times 0 + 15 \times 60 = 900$

The maximum value of Z is given by

$Z = 50 \times 10 + 15 \times 50 = 1250.$

7. Let n and p be the parameters of the distribution.
Then,
Mean = np and Variance = npq
It is given that
$n = 5$ and Mean + Variance = 1.8
$\Rightarrow \quad np + npq = 1.8 \Rightarrow \quad 5p + 5pq = 1.8$
$\Rightarrow \quad 5p + 5p(1 - p) = 1.8 \Rightarrow \quad 5p^2 - 10p + 1.8 = 0$
$\Rightarrow \quad p^2 - 2p + 0.36 = 0$
$\Rightarrow \quad (p - 0.2)(p - 1.8) = 0 \Rightarrow p = 0.2 \quad [\because \ p \neq 1]$
Thus, we have
$n = 5$, $p = 0.2$ and $q = 0.8$.
Therefore, if X denotes the binomial variate, then
$P(X = r) = {}^5C_r (0.2)^r \ (0.8)^{5-r}$, $r = 0, 1, 2, 3, 4, 5$.
This is the required binomial distribution.

8. Converting the given inequations into equations, we obtain the following equations :
$l_1 : x + 2y = 40$, $l_2 : 3x + y = 30$, $l_3 : 4x + 3y = 60$, $l_4 : x = 0$ and $l_5 : y = 0$.
The line l_1 meets the coordinate axes at $A_1(40, 0)$ and $B_1(0, 20)$ respectively.
Clearly, (0, 0) satisfies the inequation $x + 2y \leq 40$. So, the region in xy-plane that contains the origin represents the solution set of the given inequation.
The line l_2 meets coordinate axes at A_2 (10, 0) and $B_2(0, 30)$ respectively.
Clearly $O(0, 0)$ does not satisfy the inequation $3x + y \geq 30$. So the region in xy-plane which does not contain the origin is the solution set of this inequation.
The line l_3 meets coordinate axes at $A_3(15, 0)$ and $B_1(0, 20)$ respectively. Clearly $O(0, 0)$ does not satisfy the inequation $4x + 3y \geq 60$. So, the region not containing the origin in xy-plane represents the solution set of the given inequation.
Region represented by $x \geq 0$, $y \geq 0$: Clearly, the region represented by $x \geq 0$ and $y \geq 0$ is the first quadrant in xy-plane.
The shaded region A_3A_1QP in figure represents the common region of the regions represented by the above inequations. This region represents the feasible region of the given LPP.



The coordinates of the corner-points of the shaded feasible region are $A_3(15, 0)$, $A_1(40, 0)$, $Q(4, 18)$ and $P(6, 12)$. These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.
The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 20x + 10y$
$A_3(15, 0)$	$Z = 20 \times 15 + 10 \times 0 = 300$
$A_1(40, 0)$	$Z = 20 \times 40 + 10 \times 0 = 800$
$Q(4, 18)$	$Z = 20 \times 4 + 10 \times 18 = 260$
$P(6, 12)$	$Z = 20 \times 6 + 10 \times 12 = 240$

Clearly, Z is minimum at $P(6, 12)$. Hence, $x = 6$, $y = 12$ is the optimal solution of the given LPP. The optimal value of Z is 240.

9. Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:
MFS, MSF, FMS, FSM, SFM, SMF is 6.
We have,
$A = \{SMF, SFM, MFS, FMS\}$ and $B = \{MFS, SFM\}$
$\therefore \quad A \cap B = \{MFS, SFM\}$
Clearly, $n(A \cap B) = 2$ and $n(B) = 2$
$\therefore$ Required probability = $P(A/B)$

$$= \frac{n(A \cap B)}{n(B)} = \frac{2}{2} = 1$$

10. Let S be the sample space associated with the given random experiment.
Then,
$S = \{H, TH, TTH, TTTH, TTTT\}$
Let X denote the number of tosses. Then X can take values 1, 2, 3 and 4
Now,

$$P(X = 1) = P(H) = \frac{1}{2},$$

$$P(X = 2) = P(TH) = P(T)P(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4},$$

$$P(X = 3) = P(TTH) = P(T)P(T)P(H)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8},$$

and $P(X = 4) = P(TTTH \text{ or } TTTT)$

$$= P(TTTH) + P(TTTT)$$

$$\Rightarrow \quad P(X = 4) = P(T)\ P(T)\ P(T)\ P(H)$$

$$+ P(T)\ P(T)\ P(T)\ P(T) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}.$$

Thus, the probability distribution of X is given by

X	1	2	3	4
$P(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

11. Let X denote the number of hurdles knocked down by the player. Then, X follows binomial distribution with $n = 10$, $p = 1 - \frac{5}{6} = \frac{1}{6}$ and $q = \frac{5}{6}$

$\therefore \quad P(X = r) = {}^{10}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{10-r}$; $r = 0, 1, 2, \ldots, 10$

Required probability = $P(X < 2)$
$= P(X = 0) + P(X = 1)$

$= \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^9$

$= \left(\frac{5}{6}\right)^9 \left\{\frac{5}{6} + \frac{10}{6}\right\} = \left(\frac{5}{6}\right)^9 \left(\frac{15}{6}\right) = \frac{5^9 \times 5 \times 3}{6^9 \times 6} = \frac{5^{10}}{2 \times 6^9}$

12. Let X denote the number of heads in a simultaneous toss of three coins. Then, X can take values 0, 1, 2, 3.

Now, $P(X = 0) = P(TTT) = \frac{1}{8}$

$P(X = 1) = P(HTT \text{ or } TTH \text{ or } THT) = \frac{3}{8}$

$P(X = 2) = P(HHT \text{ or } THH \text{ or } HTH) = \frac{3}{8}$

and $P(X = 3) = P(HHH) = \frac{1}{8}$

Thus, the probability distribution of X is given by

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Computation of mean and variance :

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\Sigma p_i x_i = \frac{3}{2}$	$\Sigma p_i x_i^2 = 3$

We have,

$$\Sigma p_i x_i = \frac{3}{2} \text{ and } \Sigma p_i x_i^2 = 3$$

$\therefore \quad \mu = \text{Mean} = \Sigma p_i x_i = \frac{3}{2}$

and $\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

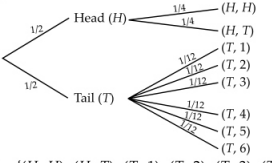
$\therefore$ Standard deviation $= \sqrt{\text{Var}(X)}$

$= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.87$

Hence, Mean $= \frac{3}{2}$, Variance $= \frac{3}{4}$ and

Standard deviation = 0.87

13. The outcomes of the experiment can be represented in the following tree diagram.
The sample space S of the experiment is given as



$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$
The probabilities of these elementary events are

$P\{(H, H)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, $P\{(H, T)\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$,

$P\{(T, 1)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$, $P\{(T, 2)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$,

$P\{(T, 3)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$, $P\{(T, 4)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$,

$P\{(T, 5)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ and, $P\{(T, 6)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Consider the following events:
A = the die shows a number greater than 4
B = there is at least one tail.
We have,
$A = \{(T, 5), (T, 6)\}$, $B = \{(H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$
and $A \cap B = \{(T, 5) (T, 6)\}$
$\because \quad P(B) = P\{(H, T)\} + P\{(T, 1)\} + P\{(T, 2)\} + P\{(T, 3)\} + P\{(T, 4)\} + P\{(T, 5)\} + P\{(T, 6)\}$

$\Rightarrow \quad P(B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$

and $P(A \cap B) = P\{(T, 5)\} + P\{(T, 6)\}$

$= \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

$\therefore \quad$ Required probability $= P(A/B)$

$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{4}} = \frac{4}{18} = \frac{2}{9}$

14. Let E_1, E_2, E_3 and A be the events defined as follows :
E_1 = the examinee guesses the answer
E_2 = the examinee copies the answer
E_3 = the examinee knows the answer
and A = the examinee answers correctly.
We have, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$

Since E_1, E_2, E_3 are mutually exclusive and exhaustive events.
$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1$
$\Rightarrow \quad P(E_3) = 1 - (P(E_1) + P(E_2))$
$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$.

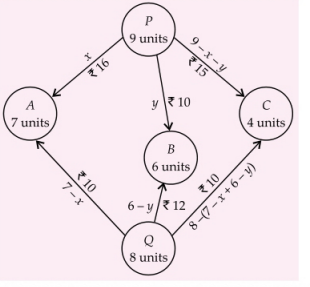
If E_1 has already occurred, then the examinee guesses. Since there are four choices out of which only one is correct, therefore the probability that he answers correctly given that he has made a guess is 1/4 *i.e.*, $P(A/E_1) = 1/4$.

It is given that $P(A/E_2) = 1/8$, and
$P(A/E_3)$ = Probability that he answers correctly given that he knew the answer
$P(A/E_3) = 1$
By Bayes' Theorem, we have
Required probability $= P(E_3/A)$

$$= \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{8} + \frac{1}{2} \times 1} = \frac{24}{29}$$

15. This problem can be explained diagramatically as follows :



Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then, $(9 - x - y)$ units will be transported from the factory at P to the depot at C.

$\therefore \quad x \geq 0$, $y \geq 0$, and $9 - x - y \geq 0 \Rightarrow x + y \leq 9$.

The weekly requirement of the depot at A is 7 units. So, $(7 - x)$ units will be transported to A from the factory at Q.

Similarly, $(6 - y)$ units will be transported to B from the factory at Q.

And, $8 - (7 - x + 6 - y) = (x + y - 5)$ units will be transported to C from the factory at Q.

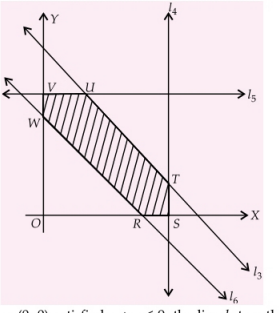
$\therefore \quad 7 - x \geq 0$, $6 - y \geq 0$ and $x + y - 5 \geq 0$

i.e., $x \leq 7$, $y \leq 6$ and $x + y \geq 5$.

The total cost of transportation is
$Z = 16x + 10(7 - x) + 10y + 12(6 - y) + 15(9 - x - y) + 10(x + y - 5)$
$\Rightarrow \quad Z = x - 7y + 227$.

Now, we have to find the values of x and y which minimizes
$Z = x - 7y + 227$, subject to the constraints
$l_1 : x \geq 0$, $l_2 : y \geq 0$, $l_3 : x + y \leq 9$, $l_4 : x \leq 7$, $y \leq 6$ and $l_5 : x + y \geq 5$.

We draw the graphs of the lines
$l_1 : x = 0$, $l_2 : y = 0$, $l_3 : x + y = 9$, $l_4 : x = 7$,
$l_5 : y = 6$ and $l_6 : x + y = 5$
as shown below.



Since (0, 0) satisfied $x + y \leq 9$, the line l_3 together with the region containing $O(0, 0)$ represents $x + y \leq 9$.

Since (0, 0) does not satisfy $x + y \geq 5$, the line l_6 together with the region not containing $O(0, 0)$ represents $x + y \geq 5$.

Since (0, 0) satisfies $x \leq 7$, the line l_4 together with the region containing $O(0, 0)$ represents $x \leq 7$.

Since (0, 0) satisfies $y \le 6$, the line l_5 together with the region containing $O(0, 0)$ represents $y \le 6$.

The y-axis and the region to its right-hand side represents $x \ge 0$.

The x-axis and the region above it represents $y \ge 0$.

Thus, the shaded region represents the feasible region whose vertices are R, S, T, U, V and W.

On solving $y = 0$ and $x + y = 5$, we get $R(5, 0)$.

On solving $y = 0$ and $x = 7$, we get $S(7, 0)$.

On solving $x = 7$ and $x + y = 9$, we get $T(7, 2)$.

On solving $x + y = 9$ and $y = 6$, we get $U(3, 6)$.

On solving $x = 0$ and $y = 6$, we get $V(0, 6)$.

On solving $x = 0$ and $x + y = 5$, we get $W(0, 5)$.

The values of $Z = x - 7y + 227$ at R, S, T, U and W are 232, 234, 220, 188, 185 and 192 respectively.

Thus, Z is minimum at $V(0, 6)$, *i.e.*, when $x = 0$ and $y = 6$.

$\therefore$ From the factory at P there must be a delivery of 0, 6 and 3 units to A, B, C respectively.

And, the factory at Q must deliver 7, 0 and 1 units to A, B, C respectively.

16. Suppose the dealer buys x fans and y sewing machines. Since the dealer has space for at most 20 items. Therefore,

$$x + y \le 20 \quad \text{...(i)}$$

A fan costs ₹ 360 and a sewing machine costs ₹ 240. Therefore, total cost of x fans and y sewing machines is ₹$(360x + 240y)$. But the dealer has only ₹ 5760 to invest.

Therefore,

$$360x + 240y \le 5760 \quad \text{...(ii)}$$

Since the dealer can sell all the items that he can buy and the profit on a fan is of ₹ 22 and on a sewing machine the profit is of ₹ 18. Therefore, total profit on selling x fans and y sewing machines is of ₹ $(22x + 18y)$.

Let Z denote the total profit. Then, $Z = 22x + 18y$.

Clearly, $x, y \ge 0$.

Thus, the mathematical formulation of the given problem is

Maximize $Z = 22x + 18y$

subject to

$x + y \le 20$

$360x + 240y \le 5760$

and, $x \ge 0$, $y \ge 0$

To solve this LPP graphically, we first convert the inequations into equations $l_1 : x + y = 20$, $l_2 : 36x + 24y = 576$, $l_3 : x = 0$, $l_4 : y = 0$. l_1 meet the coordinate axes at $A_1(20, 0)$ and $B_1(0, 20)$ respectively.

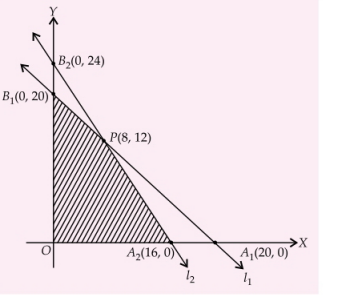
Clearly, (0, 0) satisfies the inequation $x + y \le 20$. So, the region in xy-plane that contains the origin represents the solution set of the given inequation.

The line l_2 meets axes at $A_2(16, 0)$ and $B_2(0, 24)$ respectively.

Clearly, (0, 0) satisfies the inequation $360x + 240y \le 5760$. So, the region in xy-plane that contains the origin represents the solution set of the given inequation.

Region represented by $x \ge 0$, $y \ge 0$ is the first quadrant in xy-plane. Draw the corresponding lines.

The feasible region of the LPP is shaded in figure. The corner points of the feasible region OA_2PB_1 are $O(0, 0)$, $A_2(16, 0)$, $P(8, 12)$ and $B_1(0, 20)$.



These points have been obtained by solving the corresponding intersecting lines, simultaneously.

The values of the objective function Z at corner-points of the feasible region are given in the following table.

Point(x, y)	Value of the objective function $Z = 22x + 18y$
$O(0, 0)$	$Z = 22 \times 0 + 18 \times 0 = 0$
$A_2(16, 0)$	$Z = 22 \times 16 + 18 \times 0 = 352$
$P(8, 12)$	$Z = 22 \times 8 + 18 \times 12 = 392$
$B_1(0, 20)$	$Z = 22 \times 0 + 20 \times 18 = 360$

Clearly, Z is maximum at $x = 8$ and $y = 12$. The maximum value of Z is 392.

Hence, the dealer should purchase 8 fans and 12 sewing machines to obtain the maximum profit under given conditions.

■■

mtG

Crack BITSAT with ease

The BITSAT isn't a tough exam. It is only different from the others you will appear for. The book explores everything you need to crack BITSAT like a genius!

FREE CD INSIDE with

5 Model Test Papers

Complete Theory for

Mathematics, Physics, Chemistry Logical Reasoning & English

Chapterwise MCQs for practice

1000+ MCQs in each section

5 Model Test Papers

₹525

Buy online at www.mtg.in

Available at all leading bookshops throughout India.

mtG BOOKS

For more information or to place your order :

Call at : 0124 - 4951200 @ Mail at : info@mtg.in

Log on to : www.mtg.in

ATTENTION XIth SCIENCE STUDENTS

The wait is over !

The Most Comprehensive Physics book for class XI is

NOW ON SALE !!!

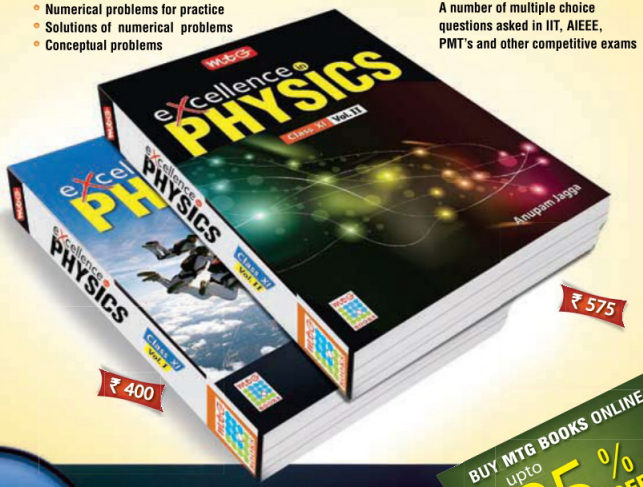
Each unit of the book is structured comprehensively such that every concept in it is described separately with

- Comprehensive theory
- Solved examples
- NCERT section
- Numerical problems for practice
- Solutions of numerical problems
- Conceptual problems

- Objective problems asked in various competitive examinations
- Short answer type problems

Long answer type problems
And each unit at the end has

A number of multiple choice questions asked in IIT, AIEEE, PMT's and other competitive exams



Available at all leading bookshops throughout India

For more information or to place your order :
0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

BUY MTG BOOKS ONLINE
upto 25% OFF
www.mtg.in

MATHS MUSING

SOLUTION SET-132

1. (c) : Mean $= \bar{x} = \dfrac{1+3+5....+2013}{1007} = 1007$

Variance, $\sigma^2 = \dfrac{1}{1007}(1^2+3^2+5^2+....+2013^2) - 1007^2 = 338016$

$\sigma = \sqrt{338016} = 581.39$

Least integer > standard deviation is 582

2. (a) : $t_n = \dfrac{1\cdot3\cdot5\cdot7...(2n-1)}{7\cdot9\cdot11...(2n+5)}$

$= 15\dfrac{1}{(2n+1)(2n+3)(2n+5)}$

$= \dfrac{15}{8}\left(\dfrac{1}{2n+1} - \dfrac{2}{2n+3} + \dfrac{1}{2n+5}\right)$

$\sum_{n=1}^{20} t_n = \dfrac{15}{8}\left(\dfrac{1}{3} - \dfrac{1}{5} - \dfrac{1}{43} + \dfrac{1}{45}\right) = \dfrac{128}{516} = \dfrac{32}{129} = \dfrac{m}{n}$

$\therefore \quad n - 4m = 1$

3. (c) : Let $H(x_1, y_1)$ be the orthocentre.

The image of H in the line $x + y = -1$ is $(-1-y_1, -1-x_1)$. It lies on the circle.

$\therefore$ Locus of H is $x^2 + y^2 + x - y - 8 = 0$

4. (b) : With zero and non-zero digit a :

$$\left(\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}\right)\binom{9}{1} = 72$$

With two non-zero digits a and b :

$$\left(\binom{4}{1} + \binom{4}{2} + \binom{4}{3}\right)\binom{9}{2} = 504$$

$\therefore$ The number of numbers is $72 + 504 = 576$

5. (d) : $\alpha + \beta + \gamma = 0, \ \alpha\beta\gamma = -\dfrac{2016}{8} = -252$

$(\alpha+\beta)^3 + (\beta+\gamma)^3 + (\gamma+\alpha)^3$

$= -(\alpha^3 + \beta^3 + \gamma^3) = -3\alpha\beta\gamma = 756$

6. (b, d): Clairaut equation

$y = xp + \sqrt{2p^2+1}, \ p = \dfrac{dy}{dx}$

Differentiate w.r.t. $x \Rightarrow \ p = p + \left(x + \dfrac{2p}{\sqrt{2p^2+1}}\right)\dfrac{dp}{dx}$

$\Rightarrow \quad \dfrac{dp}{dx} = 0, \ p = \text{constant}$

$\therefore \quad y = cx + \sqrt{2c^2+1}, c = 2 \ \Rightarrow \ y = 2x + 3$

Singular solution ; $x = \dfrac{-2p}{\sqrt{2p^2+1}}$

$y = \dfrac{1}{\sqrt{2p^2+1}}$

Eliminating $p \Rightarrow x^2 + 2y^2 = 2$

7. (a) : $A = (t_1^2, 2t_1), B(t_2^2, 2t_2)$

$t = -t_1 - \dfrac{2}{t_1} = -t_2 - \dfrac{2}{t_2}$

$\Rightarrow \quad t_1 t_2 = 2, \ t_1 + t_2 = -t$

$\therefore \quad T = (2, -t), C = (t^2, 2t)$

$\Rightarrow \quad CT^2 = (t^2+1)(t^2+4)$

8. (b) : $AB^2 = (t_1 - t_2)^2[(t_1+t_2)^2 + 4]$

$= (t^2 - 8)(t^2 + 4)$

9. (7) : $\sin x + \cos x = \dfrac{1}{4} - y, \ y = \sin x \cos x$

Squaring, $1 + 2y = \dfrac{1}{16} - \dfrac{y}{2} + y^2 \Rightarrow y = \dfrac{5}{4} - \dfrac{\sqrt{10}}{2}$

$\therefore \quad (1 - \sin x)(1 - \cos x) = \dfrac{9}{2} - \sqrt{10} - \dfrac{5}{4}$

$= \dfrac{13}{4} - \sqrt{10}, m = 13, n = 4, p = 10$

$\therefore \quad m + n - p = 7$

10. (d) :

P. $xy = \dfrac{1}{4}, x = \dfrac{t}{2}, y = \dfrac{1}{2t}$

Tangent $\dfrac{x}{2t} + \dfrac{t}{2}y = \dfrac{1}{2} \Rightarrow OA = t, OB = \dfrac{1}{t}$

$OA \cdot OB = 1$

Q. Tangent at $(\cos\theta, \sin\theta)$ is $x\cos\theta + y\sin\theta = 1$

$OA = \sec\theta, OB = \operatorname{cosec}\theta \Rightarrow \dfrac{1}{OA^2} + \dfrac{1}{OB^2} = 1$

R. Tangent at $(\cos^4\theta, \sin^4\theta)$ is

$x\sin^2\theta + y\cos^2\theta = \sin^2\theta\cos^2\theta$

$\therefore \quad OA = \cos^2\theta, OB = \sin^2\theta, OA + OB = 1$

S. Tangent at $(\cos^3\theta, \sin^3\theta)$ is

$x\sin\theta + y\cos\theta = \sin\theta\cos\theta$

$OA = \cos\theta, OB = \sin\theta, AB = 1$

Solution Sender of Maths Musing

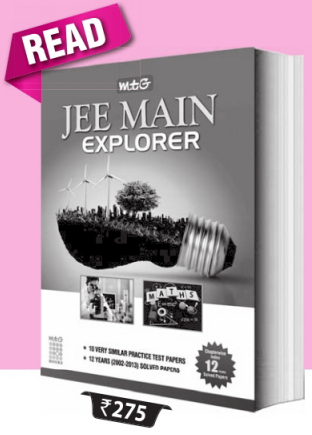
SET-132

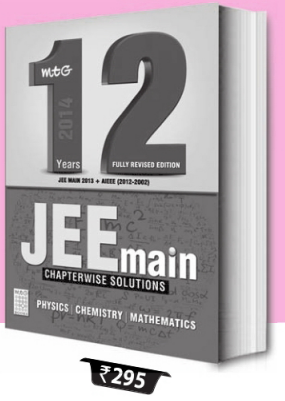
1.	Gauri Sankar Adhikary	:	Thana-Goaltore
2.	D. Hansica, N. Jayanthi, Sri Aditya Phaniram	:	Hyderabad
3.	V. Rahul	:	Warangal
4.	Soubhik Das	:	West Bengal

BEST TOOLS FOR SUCCESS IN

JEE Main

READ





10 very Similar Practice Test Papers

12 JEE MAIN 2013 + AIEEE (2012-2002)

Buy online at www.mtg.in

Available at all leading bookshops throughout India.

mtG BOOKS

For more information or to place your order :
Call at : 0124 - 4951200 @ Mail at : info@mtg.in
Log on to : www.mtg.in

YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

Q.1 Find the number of real solutions of the equation $x^2 + 4x + 7 = 2\cos x$.

– Arpitha (Kerla)

Ans. $x^2 + 4x + 7 = 2\cos x$

$\Rightarrow \quad x^2 + 4x + 4 + 3 = 2\cos x$

$\Rightarrow \quad (x+2)^2 + 3 = 2\cos x$

$\because \quad (x+2)^2 \geq 0$

$\therefore \quad (x+2)^2 + 3 \geq 3$

But $2\cos x \leq 2$ as $|\cos x| \leq 1$.

Hence there is no real solution.

Q.2 Evaluate the following integral.

$$\int (1 + x\tan x)^{-2}\,dx$$

– Sumit Nair (Cochin)

Ans. $\int (1 + x\tan x)^{-2}\,dx = \int \frac{dx}{(1 + x\tan x)^2}$

$$= \int \frac{dx}{\left(1 + \frac{x}{\cot x}\right)^2}$$

$$= \int \frac{\cot^2 x}{(x + \cot x)^2}\,dx$$

$$= \int \frac{\text{cosec}^2 x - 1}{(x + \cot x)^2}\,dx$$

Put $x + \cot x = t$

such that $(1 - \text{cosec}^2 x)dx = dt$

$$\therefore \quad \int (1 + x\tan x)^{-2}\,dx = -\int \frac{1}{t^2}\,dt$$

$$= \frac{1}{t} + c$$

$$= \frac{1}{x + \cot x} + c$$

Q.3 If a young man rides his motorcycle at 25 km/hr, he had to spend ₹ 2 per km on petrol. If he rides at a faster speed of 40 km/hr. The petrol cost increases at ₹ 5 per km. He had ₹ 100 to spend on petrol and wishes to find x is the maximum distance he can travel in one hour. Express this as an LPP and solve it graphically.

– Zarrin Khan (Lucknow)

Ans. Let the young man travels x km at 25 km/hr and y km at 40 km/hr, then LPP can be stated as

Maximize: $Z = x + y$

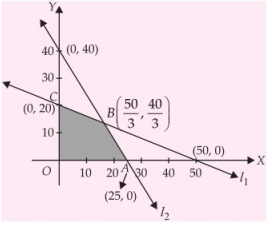
Subject to the constraints :

$$2x + 5y \leq 100 \Rightarrow l_1 : 2x + 5y = 100 \quad \ldots(1)$$

$$\frac{x}{25} + \frac{y}{40} \leq 1 \Rightarrow l_2 : 8x + 5y = 200 \quad \ldots(2)$$

$x, y \geq 0$

To solve the LPP graphically, we first convert inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in the figure.



The coordinates of the corner points of the feasible region $OABC$ are $O(0, 0)$, $A(25, 0)$, $B\left(\frac{50}{3}, \frac{40}{3}\right)$ and $C(0, 20)$

The value of objective function at these points is given in the table :

Point	Value of objective function, $Z = x + y$
$O(0, 0)$	$Z = 0 + 0 = 0$ km
$A(25, 0)$	$Z = 25 + 0 = 25$ km
$B\left(\frac{50}{3}, \frac{40}{3}\right)$	$Z = \frac{50}{3} + \frac{40}{3} = 30$ km
$C(0, 20)$	$Z = 0 + 20 = 20$ km

Clearly, Z is maximum at $x = \frac{50}{3}$, $y = \frac{40}{3}$.

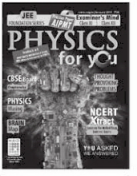
Thus the maximum distance travelled in 1 hr is 30 km travelling $\frac{50}{3}$ km at 25 km/hr and $\frac{40}{3}$ km at 40 km/hr.

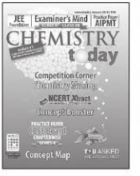
■■

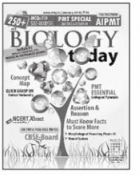
mtG SUPER SUBSCRIPTION OFFER

NEW!

JEE(Main & Advanced) | AIPMT | PMTs | State level Engg,/Medical Entrance Exams







- Predict future paper trends with India's best teaching brains preparing Very-Similar Papers every month to give student's practice, practice and more practice.
- These magazines are the only source for Solved Test Papers of all the National and State level Engg./Medical Entrance Exams.
- Articles are developed by experts essentially to strengthen your conceptual understanding.
- Moreover, we will keep you informed about latest exam dates, trends, syllabus, new study techniques, time management skills and much more.
- Plus, confidence building exercises with Self Tests and success stories of IIT/PMT toppers.
- Also elevate you to the International level with Olympiad/Contest Problems. And much more.

SUBSCRIBE TODAY! Save ₹ 2220 on the cover price

Nobody will blame you if you don't subscribe these magazines. And chances are you're good enough to make it on your own. But does it make sense to take a chance when so much is at stake? Each month, these magazines will keep you ahead of the competition. That too delivered at your doorstep!

For any information Call
08800255334
08800255335

A little effort now could ensure you a place in prestigious Engineering / Medical Institute.

Fill this form and send it to us.

SUBSCRIPTION FORM

☑ **YES !** I am interested in (please ✓tick the box)

BEST OFFER

PCMB subscription
☐ 1 yr : ₹ 900 ☐ 2 yr : ₹ 1500 ☐ 3 yr : ₹ 2100

PCM subscription
☐ 1 yr : ₹ 800 ☐ 2 yr : ₹ 1200 ☐ 3 yr : ₹ 1700

PCB subscription
☐ 1 yr : ₹ 800 ☐ 2 yr : ₹ 1200 ☐ 3 yr : ₹ 1700

Single Magazine Subscription
☑ *Tick any one :* ☐ MT ☐ CT ☐ PFY ☐ BT
☐ 1 yr : ₹ 300 ☐ 2 yr : ₹ 500 ☐ 3 yr : ₹ 675

Name : .....
Address : .....
.....
..... Pin
Phone /Mobile .....
E-mail : .....

Send Demand Draft/Money Order in favour of **MTG Learning Media (P) Ltd.,** payable at New Delhi.

Note : You can start the subscription from any month of the year for a period of 12 months (1year) or 24 months (2 years)or 36 months (3 years).

* Magazines are despatched by book-post by the 4th day of every month, each magazine separately. However, if any subscriber wants the magazines to reach every month by courier or registered post, additional charges applicable are as follows:

Rs. 150 for 1 year subscription | Rs. 250 for 2 years subscription | Rs. 350 for 3 years subscription

Fill in the form and send it to Subscription Dept., MTG, Plot 99, Sector 44, Gurgaon -122003 (HR.) Tel.: 08800255334 / 35, E-mail: info@mtg.in or Subscribe online at www.mtg.in